EXPONENTIAL-FAMILY RANDOM GRAPH MODELS FOR RANK-ORDER RELATIONAL DATA

Pavel N. Krivitsky*
Carter T. Butts[†]

Abstract

Rank-order relational data, in which each actor ranks other actors according to some criterion, often arise from sociometric measurements of judgment or preference. The authors propose a general framework for representing such data, define a class of exponential-family models for rank-order relational structure, and derive sufficient statistics for interdependent ordinal judgments that do not require the assumption of comparability across raters. These statistics allow estimation of effects for a variety of plausible mechanisms governing rank structure, both in a cross-sectional context and evolving over time. The authors apply this framework to model the evolution of liking judgments in an acquaintance process and to model recall of relative volume of interpersonal interaction among members of a technology education program.

*University of Wollongong, Wollongong, New South Wales, Australia
[†]University of California, Irvine, Irvine, CA, USA

Corresponding Author:
Pavel N. Krivitsky, School of Mathematics and Applied Statistics, University of Wollongong, Wollongong, NSW, Australia
Email: pavel@uow.edu.au

1. INTRODUCTION

Rank-order sociometric data, in which each actor in a network ranks other actors according to some criterion, have a long history in the social sciences. Among many other instances, Sampson (1968) famously asked each of 18 novitiates in a monastery to rank his three most liked novitiates among the other 17; Newcomb (1961) measured evolving rankings of one another by 17 men living in fraternity-style housing over the course of a semester; and, more recently, Wave I of the National Longitudinal Study of Adolescent Health asked high school students to list, in order, up to five male and up to five female friends (Harris et al. 2003). Although many network processes (e.g., diffusion, brokerage, exchange) are only sensibly posited for networks with categorical- or ratio-scale relationship states, many others, particularly those involving personal preferences (e.g., liking, advice seeking), are much more readily represented ordinally and, indeed, may not even have interval, ratio, or categorical meaning across raters. This can be true even when data are not collected in an explicitly ordinal fashion. For instance, Johnson, Boster, and Palinkas (2003) asked personnel in an isolated environment (the Amundsen-Scott South Pole Station) to rate their degree of interaction with one another on a scale ranging from 0 to 10, with 0 indicating no interaction and 10 indicating a "great deal" of interaction. Although it is reasonable to assume that such ratings are ordinally coherent within rater (e.g., if Bob rates Sally below Jill, then Bob regards himself as interacting more with Jill than with Sally), such ratings cannot be compared *across* raters: if Bob rates his interaction with Jill at 4, and Sally rates her interaction with Jill at 6, we have no basis for concluding that Sally's interaction with Jill is stronger than Bob's. Such interaction rating data are thus "local" to the rater and must be analyzed in a manner that avoids cross-rater comparisons.

The most common approach taken to analyzing rank-valued network data in current practice is to dichotomize ranks into binary ties, defining a tie to be present if a given ego had ranked a given alter above a certain cutoff and absent otherwise. Many methods of dichotomizing have been proposed. For instance, cutoffs have been set at a particular rank (such as top five) (e.g., Arabie, Boorman, and Levitt 1978; Breiger, Boorman, and Arabie 1975; Harris et al. 2003; Pattison 1982; Wasserman 1980; White, Boorman, and Breiger 1976), set at a particular quantile (such as top 50 percent) (Krackhardt and Handcock 2007), or found adaptively

(Doreian et al. 1996). Another common approach is to focus on rank correlations and on treating ranks on an additive scale (Newcomb 1956; Nakao and Romney 1993).

These approaches come with significant limitations. Dichotomizing ties requires a threshold point to be selected, inevitably discarding information and possibly introducing biases (Thomas and Blitzstein 2011), while methods such as rank correlation are limited to simple comparisons and cannot, for example, be used to examine the strength of one social factor after controlling for the effects of another. More importantly, these techniques implicitly assume that tie values can be equated across raters, an assumption that is often unjustified. When the presence of an (i,j) tie has a different empirical meaning than the presence of a (k,j) tie, conventional network analytic techniques (such as centrality indices) may prove misleading.

Modeling frameworks explicitly designed for rank-order data would address these limitations, but to date work on model-based approaches to rank-order network data has been very limited. Gormley and Murphy (2008), for instance, used a generalization of the Plackett-Luce model (Plackett 1975) in a latent position framework to model what can be viewed as a bipartite rank-order network of affiliations from voters to candidates in Irish proportional representation through the single transferable vote elections. Null models for comparison of rank-order (or otherwise valued) data structures were developed by Hubert (1987), and model-based extensions of this approach for comparison of multiple structures were introduced by Butts (2007b). (See also related work on null hypothesis testing in a network regression context, such as that of Krackhardt 1987 and Dekker, Krackhardt, and Snijders 2007.) This latter work is focused on modeling degrees of correspondence *between* relational structures, and it does not attempt to model the internal properties of rank-valued networks themselves. This second problem is the focus of the present article.

For modeling of internal network structure, exponential-family random graph models (ERGMs) or p^* models (Holland and Leinhardt 1981; Robins, Pattison, and Wasserman 1999; Wasserman and Pattison 1996) are the currently favored approach. Models parameterized in this way have been applied to social network data in a variety of contexts, including dichotomized rank-order data (Goodreau, Kitts, and Morris 2008a; Krackhardt and Handcock 2007); used in this latter capacity, they inherit the difficulties with dichotomizing noted above. Robins

et al. (1999) were the first to introduce a systematic treatment of ERGMs for categorically valued network data, along with procedures for approximate inference using pseudo-likelihood estimation. The model families they propose can be used when edge values are ordinal in an absolute sense but assume (1) that values can be meaningfully equated across raters and (2) that there is a well-defined zero value indicating the absence of a tie, that is qualitatively different from other possible tie values, and, per (1), equivalent across raters. Snijders (1996) applied the stochastic actor-oriented framework to Newcomb's (1961) data as well but made similar assumptions, also treating the ranks on an interval scale.

Building on this work, Krivitsky (2012) formulated a generalized framework for exponential-family models on networks whose ties have values (categorical or otherwise) and introduced Markov-chain Monte Carlo (MCMC) methods for simulation and maximum likelihood inference in this more general case. The Krivitsky formulation provides a basis for generalizing to models of the kind we consider here, but it retains the assumption of "absolute" edge values that have a constant meaning across raters. In this paper, we develop ERGMs for "locally" ordinal relational data in which ratings cannot be directly compared across subjects; we focus on the foundational case of complete rankings but introduce model terms that can be used with more general models (such as those for partial orders). In Section 2, we discuss representation of ordinal relational data and introduce the probabilistic framework for exponential-family models for them. In Section 3, we describe statistics that can be used to model common network properties within this framework. Two applications of this framework are demonstrated in Section 4, and some extensions of the framework are discussed in Section 5. Additional details involved in simulation and inference on these models are provided in Appendix A.

2. EXPONENTIAL-FAMILY FRAMEWORK FOR ORDINAL RELATIONAL DATA

2.1. *Actors, Rankings, and Comparisons*

We begin this section by defining notation for representation of ordinal relational data and by establishing basic principles for using such data in a manner that respects their intrinsic measurement properties.

Consider a set of n actors, N, whom we index as $N = \{A, B, C, \ldots\}$. Each actor in N will be a rater in our network of interest; except as noted otherwise, the objects being rated by each rater are the other members of N (though we will consider other possibilities in Section 5.1). For various purposes, it will be helpful to have specific notation to refer to the set of possible p-tuples of distinct actors in N; we refer to this as the pth "distinct Cartesian power" of N. Recall that the pth (ordinary) Cartesian power of a set N is the set of all ordered p-tuples of individuals from N. For example, if $N = \{A, B, C\}$, then the second Cartesian power of N is $N^2 = \{(A, A), (A, B), (A, C), (B, A), (B, B), (B, C), (C, A), (C, B), (C, C)\}$. By analogy, we denote the pth distinct Cartesian power of N (the set containing all p-tuples of N whose elements do not repeat) by $N^{p\neq} \subset N^p$. Following our example, then, the second *distinct* Cartesian power of N is given by $N^{2\neq} = \{(A, A), (A, B), (A, C), (B, A), (B, B), (B, C), (C, A), (C, B), (C, C)\}$.[1]

As discussed in the introduction, our data consist of observations in which each actor (*ego*) $i \in N$ in the network provides some ranking or ordering of the other actors (*alters*), and, possibly, of himself or herself. In other words, each actor i defines an ordinal relation $\succ_i$ over set N. This relation could represent "preferred to," "interacted more with than," "judged to be taller than," or any other judgment of interest. Importantly, we note that for two egos i and j, $\succ_i$ need not equal $\succ_j$, and the ratings of some alter k by i and j cannot be compared directly; we may, however, meaningfully ask, for example, whether $(k \succ_i l) = (k \succ_j l)$—whether i and j's rankings of k and l are concordant, and our modeling framework is founded on exactly these distinctions.

A simple example of a ranking structure is provided in Figure 1, in which actor A (ego) ranks alter D above B and C and C above B. In our above notation, this corresponds to $D \succ_A C$, $C \succ_A B$, and $D \succ_A B$. The presence of corresponding structures associated with Egos B, C, and D (not illustrated, but shown in the rank matrix of Figure 1, right) results in a complete ordinal network.

In general, we make few assumptions regarding $\succ_i$. Assuming that ego may not include self in the ranking, we require that for all ego-alter-alter triples $(i, j, k) \in N^{3\neq}$, i reports either $j \succ_i k$ or $j \nsucc_i k$. With the additional assumption of transitivity ($j \succ_i k \wedge k \succ_i l \Rightarrow j \succ_i l$), the above formulation leads to a partial ordering of alters by each ego. If incomparability is also transitive ($j \nsucc_i k \wedge k \nsucc_i l \Rightarrow j \nsucc_i l$), a weak ordering results, which may be used to represent rank data in which
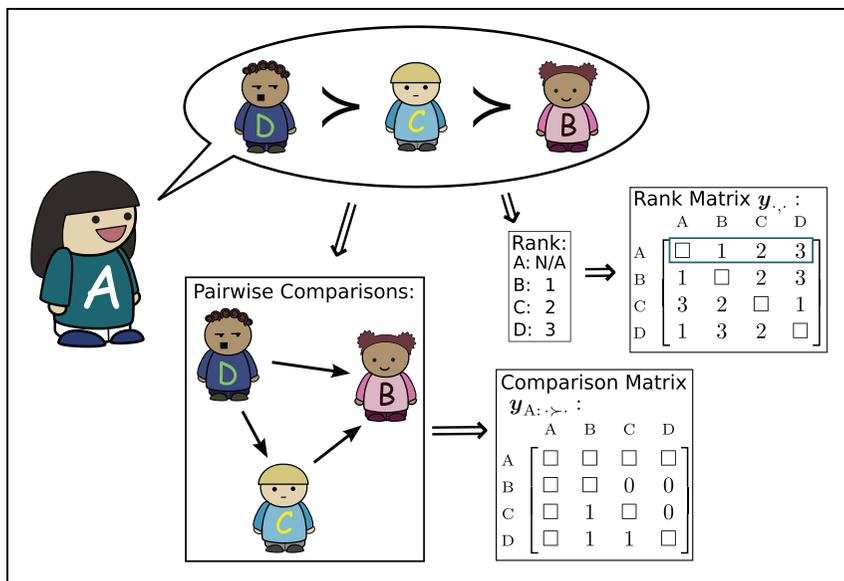
Figure 1. Ego A's report regarding her ranking of B, C, and D can be encoded as a rank ordering or as pairwise comparisons. Here, Ego A's response ranks D highest, then C, then B. We may encode this report by assigning a rank of 1 to B, 2 to C, and 3 to D, resulting in a row of rank matrix $y_{\cdot,\cdot}$; or we may consider all pairwise comparisons implied by A: that D is ranked over C, D is ranked over B, and C is ranked over B, and the opposite does not hold, resulting in a binary matrix of comparison indicators $y_{A:\cdot\succ\cdot}$. Here, boxes (□) denote matrix entries that are unobservable and/or meaningless: in this case, these include A's row and column entries in the comparison matrix and the diagonal of the rank matrix, because A is not permitted to rank herself among the others; and the diagonal of the comparison matrix, because it is meaningless to compare an alter with himself or herself.

Note: Images of characters are based (with modifications) on public domain clip art from Open Clipart.org by user anarres (http://www.openclipart.org/user-detail/anarres).

"ties" are allowed. Finally, a further constraint on $\succ_i$ that $j \not\succ_i k \Rightarrow k \succ_i j$ results in a complete ordering, which may arise when an ego is forced to rank all of the alters with no equal ranks permitted. This is an important special case, and we consider it here in more detail.

We will, furthermore, focus on the case in which the ego does not report a ranking for self and where the set of egos is the same as the set

of alters (e.g., people ranking other people who rank them in turn, as opposed to consumers ranking brands or other objects), so relation $j \succ_i k$ is meaningful only for $(i,j,k) \in N^{3\neq}$. If an ego is permitted to rank itself among the others, it is also possible for j or k (but not both) to equal i: $(i,j,k) \in N \times N^{2\neq}$, and if we call the set of people in the people-ranking-objects scenario N_e and the set of objects N_a, then $(i,j,k) \in N_e \times N_a^{2\neq}$. This last case is discussed further in Section 5.1.

Rankings are not assumed to be comparable across egos: for each ego i, we may only say that one alter is "$\succ_i$" another (or that this does not hold). To concisely represent this fundamental operation in a specific ordinal network y, we define an indicator

$$y_{i:j\succ k} \equiv \begin{cases} 1 & \text{if } j \succ_i k \text{ i.e., } i \text{ ranks } j \text{ above } k; \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

This represents the basic distinction that can be unequivocally made from locally ordinal data. As we shall see in Section 3, being limited to such statements does not prevent us from specifying a very rich class of models.

2.2. *Representations of Ordinal Networks*

We make use of two numerical representations of the observed networks of rankings, which we illustrate on toy networks with $n = 4$: a network of complete orderings y in Figure 2a and a network of idiosyncratic ranking structures $y^\star$ in Figure 3. Figure 2 also shows representations of two perturbations: (b) and (c) of y. We make use of them in subsequent sections on model terms and interpretations.

First, a network of a complete or a weak ordering y may be encoded simply as a row in an $n \times n$-matrix of ranks that we denote $y_{.,.}$, with $y_{i,j}$ being the rank by ego i of alter j. Consider the ranking structure in network y in Figure 2a. As illustrated in Figure 1, the ranking reported by Ego A can be expressed by assigning the highest possible rank $n - 1 = 3$ to D, the highest ranked alter, $n - 2 = 2$, to C, the alter with the next highest ranking, and 1 to B, the actor with the lowest ranking. This representation is slightly misleading, in that these ranks are not comparable across egos (rows). If the ordering is weak (i.e., there are ties), alters may share ranks, as with Ego A ranking Alters B and C equally in the idiosyncratic network $y^\star$ in Figure 3. The diagonal of $y_{.,.}$ is undefined,

y Ranking:

A : D ≻ C ≻ B
B : D ≻ C ≻ A
C : A ≻ B ≻ D
D : B ≻ C ≻ A

$y_{\cdot,\cdot}$

$$\begin{bmatrix} \square & 1 & 2 & 3 \\ 1 & \square & 2 & 3 \\ 3 & 2 & \square & 1 \\ 1 & 3 & 2 & \square \end{bmatrix}$$

$y_{A:\cdot\succ\cdot}$

$$\begin{bmatrix} \square & \square & \square & \square \\ \square & \square & 0 & 0 \\ \square & 1 & \square & 0 \\ \square & 1 & 1 & \square \end{bmatrix}$$

$y_{B:\cdot\succ\cdot}$

$$\begin{bmatrix} \square & \square & 0 & 0 \\ \square & \square & \square & \square \\ 1 & \square & \square & 0 \\ 1 & \square & 1 & \square \end{bmatrix}$$

$y_{C:\cdot\succ\cdot}$

$$\begin{bmatrix} \square & 1 & \square & 1 \\ 0 & \square & \square & 1 \\ \square & \square & \square & \square \\ 0 & 0 & \square & \square \end{bmatrix}$$

$y_{D:\cdot\succ\cdot}$

$$\begin{bmatrix} \square & 0 & 0 & \square \\ 1 & \square & 1 & \square \\ 1 & 0 & \square & \square \\ \square & \square & \square & \square \end{bmatrix}$$

(a) y: original network

$y^{A:B\rightleftharpoons C}$ Ranking:

A : D ≻ B ≻ C
B : D ≻ C ≻ A
C : A ≻ B ≻ D
D : B ≻ C ≻ A

$y^{A:B\rightleftharpoons C}_{\cdot,\cdot}$

$$\begin{bmatrix} \square & 2 & 1 & 3 \\ 1 & \square & 2 & 3 \\ 3 & 2 & \square & 1 \\ 1 & 3 & 2 & \square \end{bmatrix}$$

$y^{A:B\rightleftharpoons C}_{A:\cdot\succ\cdot}$

$$\begin{bmatrix} \square & \square & \square & \square \\ \square & \square & 1 & 0 \\ \square & 0 & \square & 0 \\ \square & 1 & 1 & \square \end{bmatrix}$$

(b) $y^{A:B\rightleftharpoons C}$: Ego A swapped B and C

$y^{A:B\rightleftharpoons D}$ Ranking:

A : B ≻ C ≻ D
B : D ≻ C ≻ A
C : A ≻ B ≻ D
D : B ≻ C ≻ A

$y^{A:B\rightleftharpoons D}_{\cdot,\cdot}$

$$\begin{bmatrix} \square & 3 & 2 & 1 \\ 1 & \square & 2 & 3 \\ 3 & 2 & \square & 1 \\ 1 & 3 & 2 & \square \end{bmatrix}$$

$y^{A:B\rightleftharpoons D}_{A:\cdot\succ\cdot}$

$$\begin{bmatrix} \square & \square & \square & \square \\ \square & \square & 1 & 1 \\ \square & 0 & \square & 1 \\ \square & 0 & 0 & \square \end{bmatrix}$$

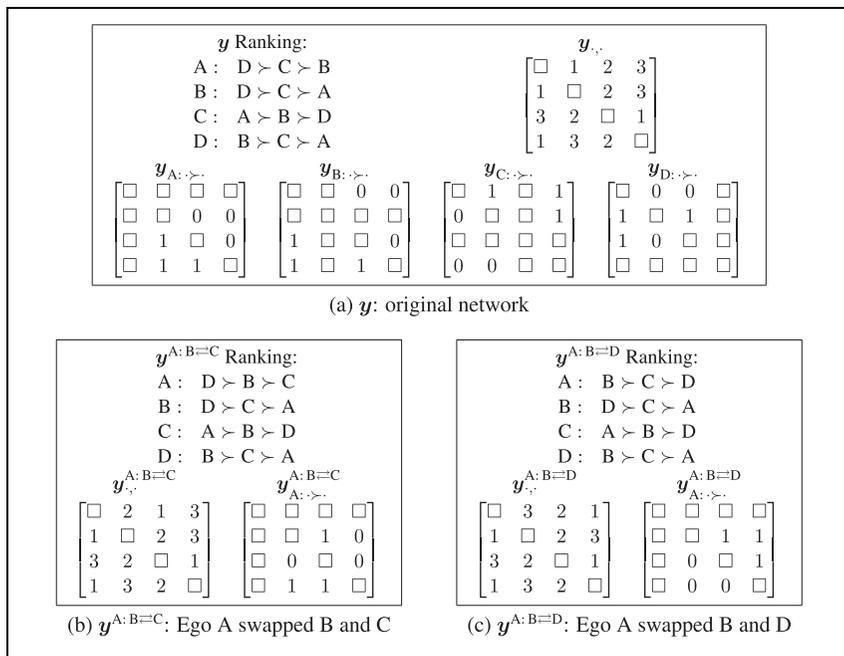(c) $y^{A:B\rightleftharpoons D}$: Ego A swapped B and D

Figure 2. Representations of complete rankings y and illustration of $y^{i:j\rightleftharpoons k}$ (defined in Section 3.1) for $n = 4$. Here, boxes ($\square$) denote matrix entries that are unobservable and/or meaningless. In $y^{A:B\rightleftharpoons C}$(b) and $y^{A:B\rightleftharpoons D}$(c), pairwise comparison matrices $y_{i:\cdot\succ\cdot}$ for i other than A are unchanged from those of y (a).

unless the egos also rank themselves in the data as Egos B and C do in Figure 3.

Second, we may represent the comparisons reported by an ego i, or implied by i's ranking, in a binary $n \times n$-matrix of pairwise comparisons, which we denote $\mathbf{y}_{i:\cdot\succ\cdot}$. Figure 1 shows how the reported complete ranking can be thus encoded: because D is ranked over both B and C, the corresponding elements in the matrix $y_{A:D\succ B} = y_{A:D\succ C} = 1$, while C is not ranked over D, so $y_{A:D\succ C} = 0$. Because A does not rank self, the row and column corresponding to A are undefined, and because it is not meaningful to compare an alter with itself, so is the diagonal. (Notably, if A were allowed to rank self, A's row and column would be defined, and if A had then *chosen* to not rank self, they would be set to 0.) The collection of reported rankings by all the egos in N can then be combined into a binary $n \times n \times n$-array.
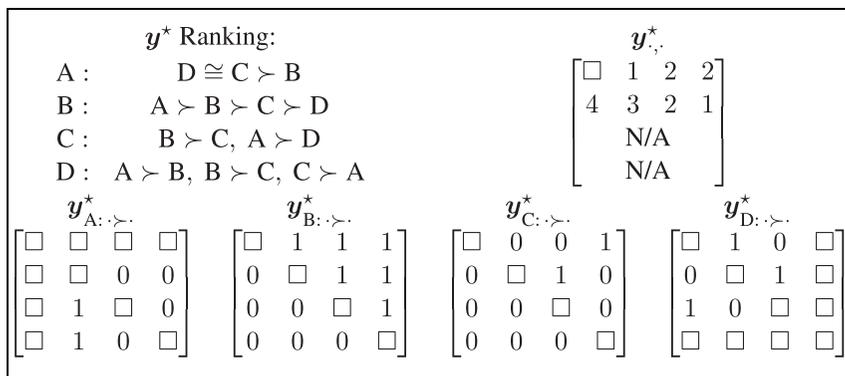
$$
\begin{array}{ll}
& \boldsymbol{y}^{\star} \text{ Ranking:} \\
A: & D \cong C \succ B \\
B: & A \succ B \succ C \succ D \\
C: & B \succ C,\ A \succ D \\
D: & A \succ B,\ B \succ C,\ C \succ A
\end{array}
\qquad
\boldsymbol{y}^{\star}_{:,.}
\begin{bmatrix}
\square & 1 & 2 & 2 \\
4 & 3 & 2 & 1 \\
& \text{N/A} & & \\
& \text{N/A} & &
\end{bmatrix}
$$

$$
\boldsymbol{y}^{\star}_{A:\,\cdot\,\succ\cdot}
\begin{bmatrix}
\square & \square & \square & \square \\
\square & \square & 0 & 0 \\
\square & 1 & \square & 0 \\
\square & 1 & 0 & \square
\end{bmatrix}
\quad
\boldsymbol{y}^{\star}_{B:\,\cdot\,\succ\cdot}
\begin{bmatrix}
\square & 1 & 1 & 1 \\
0 & \square & 1 & 1 \\
0 & 0 & \square & 1 \\
0 & 0 & 0 & \square
\end{bmatrix}
\quad
\boldsymbol{y}^{\star}_{C:\,\cdot\,\succ\cdot}
\begin{bmatrix}
\square & 0 & 0 & 1 \\
0 & \square & 1 & 0 \\
0 & 0 & \square & 0 \\
0 & 0 & 0 & \square
\end{bmatrix}
\quad
\boldsymbol{y}^{\star}_{D:\,\cdot\,\succ\cdot}
\begin{bmatrix}
\square & 1 & 0 & \square \\
0 & \square & 1 & \square \\
1 & 0 & \square & \square \\
\square & \square & \square & \square
\end{bmatrix}
$$

Figure 3. Representations of idiosyncratic ordering structures for $n = 4$. Here, boxes ($\square$) denote matrix entries that are unobservable and/or meaningless. Ego A does not rank C or D over one another but ranks both above B, for a partial ordering; Ego B is permitted to rank self along with the others; Ego C is permitted to rank self, but it does not establish a weak order, since sets $\{A, D\}$ and $\{B, C\}$ are incomparable; and Ego D reports comparisons that violate transitivity. Notice that not all these can be represented using a rank matrix $\boldsymbol{y}^{\star}_{:,.}$, but all can be represented using the comparison operation $y_{:\,\cdot\,\succ\cdot}$ and sample space constraints.

As we have noted, because the framework itself requires relatively few assumptions regarding $\succ_i$, pairwise comparison matrices can encode a wider variety of ranking structures: For example, in $\boldsymbol{y}^{\star}$ in Figure 3, Ego C does not provide enough information to establish a weak order, while Ego D's reports violate transitivity of comparisons. This precludes their representation in the corresponding rows of $\boldsymbol{y}^{\star}_{:,.}$ but not their representation as $\boldsymbol{y}^{\star}_{C:\,\cdot\,\succ\cdot}$ and $\boldsymbol{y}^{\star}_{D:\,\cdot\,\succ\cdot}$.

2.3. *Model Formulation and Specification for ERGMs for Compete Rankings*

Krivitsky (2012) suggested that a sample space of complete rankings of every actor in a network by every other actor can be represented by a directed network with no self-loops, whose set of observed relations $\mathbb{Y} = N^{2\neq}$ maps to dyad values $\mathbb{S} = \{1 .. n - 1\}$, with the ranked nature of the data leading to a complex constraint: that an ego i *must* assign a unique rank to each possible alter. Formally,

$$\mathcal{Y} = \{\mathbf{y}' \in \mathbb{S}^{\mathbb{Y}} : \forall_{i \in N} \forall_{r \in \mathbb{S}} \exists!_{j \in N \setminus \{i\}} y'_{i,j} = r\}. \tag{2}$$

Again, this representation is slightly misleading in that elements of $\mathbb{S}$ have only ordinal and not interval or ratio meanings, and, as we noted above, they are only ordered within the rankings of a given ego. That is, it makes sense to ask if for some $\mathbf{y} \in \mathcal{Y}$ and $(i,j,k) \in N^{3\neq}$, $y_{i,j} > y_{i,k}$—that is, whether i ranks j over k—but not to evaluate the difference between ranks $(y_{i,j} - y_{i,k})$ or to compare ranks by different egos $(y_{j,i} > y_{k,i})$. It does, however, represent distinct complete rankings in a concise and convenient manner, so we make use of it, with the proviso that the statistics evaluated on $\mathbf{y} \in \mathcal{Y}$ make use of no operation other than comparison within an ego's rankings $y_{i:j \succ k}$. (Equation 1).

Taking the set defined by equation (2) as our sample space, we can specify an exponential family for rank-order networks by defining a sufficient statistic $\mathbf{g}(\mathbf{y}; \mathbf{x})$, a function of a network $\mathbf{y} \in \mathcal{Y}$ that may also depend on exogenous covariates $\mathbf{x} \in \mathbb{X}$ (assumed fixed and known), for an exponential family and parametrized by $\boldsymbol{\theta} \in \mathbb{R}^p$. The probability associated with each network $\mathbf{y}$ in the sample space is then

$$\Pr_{\boldsymbol{\theta};\mathbf{g},\mathbf{x}}(\mathbf{Y} = \mathbf{y}) = \frac{\exp\{\boldsymbol{\theta} \cdot \mathbf{g}(\mathbf{y}; \mathbf{x})\}}{\kappa_{\mathbf{g},\mathbf{x}}(\boldsymbol{\theta})}, \mathbf{y} \in \mathcal{Y} \tag{3}$$

with the normalizing factor

$$\kappa_{\mathbf{g},\mathbf{x}}(\boldsymbol{\theta}) = \sum_{\mathbf{y}' \in \mathcal{Y}} \exp\{\boldsymbol{\theta} \cdot \mathbf{g}(\mathbf{y}'; \mathbf{x})\}. \tag{4}$$

This is an exact parallel to the more familiar ERGMs for dichotomous data (e.g., Wasserman and Pattison 1996). For notational convenience, we will drop $\mathbf{x}$ from now on, unless the term in question uses it explicitly.

3. TERMS AND PARAMETERS FOR ORDINAL RELATIONAL DATA

We now introduce and discuss a variety of sufficient statistics $\mathbf{g}(\cdot; \cdot)$ for the model of equation (3) ("model terms") that abide by the restrictions discussed in Section 2 while viably representing phenomena frequently observed in social networks. For each term, we discuss how it may be

interpreted without assuming rankings to have more than ordinal meaning; numerical examples are provided in Appendix B.

3.1. *Interpreting Model Terms Using "Promotion" Statistics*

For binary ERGMs, Snijders et al. (2006), Hunter, Handcock, et al. (2008), and others have used *change statistics* or *change scores*, the effect on the model of a toggling of a tie—an atomic change in the binary network structure—to aid in interpreting the model terms. For complete ordering data, an ego changing the ranking of one alter necessarily changes the ranking of at least one other, and the atomic change—one that affects the fewest alters—is to swap the rankings of two who are adjacently ranked. We thus use the effect of having ego i "promote" a promotable alter $j \in \{k \in N : k \neq i \wedge y_{i,k} < n - 1\}$, swapping j's rank with that of the alter previously ranked immediately above j, as such a change. When it is clear from the context (as it is below), we will use j^+ to refer to the promoted-over alter. (See Butts 2007b for a related use of pairwise permutations to assess model terms.)

Let $y^{i:j\rightleftarrows j^+}$ represent the network y with i's ranking of j and j^+, the actor previously ranked by i immediately above j, swapped. We define a *promotion statistic* as

$$\Delta_{i,j}^{\nearrow} g(y) \equiv g(y^{i:j\rightleftarrows j^+}) - g(y),$$

that is, the change in g resulting from i "promoting" j by one rank (and demoting the alter above him or her). An example of this is given in Figure 2: starting with a network y (a), Ego A promotes Alter B (who had been ranked immediately below C in y) over C to create $y^{A:B\rightleftarrows C}$(b). Its effect on $y_{.,.}$ is to swap the rank values $y_{A,B}$ and $y_{A,C}$, and the effect on the pairwise comparison matrix $y_{A:.\succ.}$ is the smallest possible, subject to the complete ordering constraint: the indicators $y_{A:B\succ C}$ and $y_{A:C\succ B}$ are swapped. In contrast, if Ego A promotes Alter B over D, who had not been adjacently ranked to B in y, producing $y^{A:B\rightleftarrows D}$(c), the effect on $y_{.,.}$ is similar to before, but the effect on $y_{A:.\succ.}$ is profound.

Analogously to change scores, the promotion statistic emerges when considering the conditional probability of an ego i ranking an alter j over alter j^+, other rankings being the same. To see how, let y be a network that has i ranking $k \equiv j^+$ immediately over j, so that $y^{i:j\rightleftarrows k}$ is an otherwise identical network where i ranks j over k instead. Then,

$$\Pr{}_{\boldsymbol{\theta};\boldsymbol{g}}(y_{i:j\succ k}=1\,|\,\boldsymbol{Y}=\boldsymbol{y}^{i:j\rightleftarrows k}\vee \boldsymbol{Y}=\boldsymbol{y})$$

$$=\frac{\Pr{}_{\boldsymbol{\theta};\boldsymbol{g}}\{Y_{i:j\succ k}=1\wedge(\boldsymbol{Y}=\boldsymbol{y}^{i:j\rightleftarrows k}\vee \boldsymbol{Y}=\boldsymbol{y})\}}{\Pr{}_{\boldsymbol{\theta};\boldsymbol{g}}(\boldsymbol{Y}=\boldsymbol{y}^{i:j\rightleftarrows k}\vee \boldsymbol{Y}=\boldsymbol{y})}$$

$$=\frac{\Pr{}_{\boldsymbol{\theta};\boldsymbol{g}}(\boldsymbol{Y}=\boldsymbol{y}^{i:j\rightleftarrows k})}{\Pr{}_{\boldsymbol{\theta};\boldsymbol{g}}(\boldsymbol{Y}=\boldsymbol{y}^{i:j\rightleftarrows k})+\Pr{}_{\boldsymbol{\theta};\boldsymbol{g}}(\boldsymbol{Y}=\boldsymbol{y})}$$

$$=\frac{\exp\{\boldsymbol{\theta}\cdot \boldsymbol{g}(\boldsymbol{y}^{i:j\rightleftarrows k})\}/\kappa_{\boldsymbol{g}}(\boldsymbol{\theta})}{\exp\{\boldsymbol{\theta}\cdot \boldsymbol{g}(\boldsymbol{y}^{i:j\rightleftarrows k})\}/\kappa_{\boldsymbol{g}}(\boldsymbol{\theta})+\exp\{\boldsymbol{\theta}\cdot \boldsymbol{g}(\boldsymbol{y})\}/\kappa_{\boldsymbol{g}}(\boldsymbol{\theta})}$$

$$=\frac{\exp\left(\boldsymbol{\theta}\cdot\{\boldsymbol{g}(\boldsymbol{y}^{i:j\rightleftarrows k})-\boldsymbol{g}(\boldsymbol{y})\}\right)}{\exp\left[\boldsymbol{\theta}\cdot\left\{\boldsymbol{g}(\boldsymbol{y}^{i:j\rightleftarrows k})-\boldsymbol{g}(\boldsymbol{y})\right\}\right]+1}$$

$$=\operatorname{logit}^{-1}\{\boldsymbol{\theta}\cdot \Delta_{i,j}^{\diagup}\boldsymbol{g}(\boldsymbol{y})\},$$

because, by construction, $k\equiv j^{+}$ in $\boldsymbol{y}$. Similarly, we may consider the conditional odds of i ranking j over $k\equiv j^{+}$ or the ratio between its probability and that of an otherwise identical network where k outranks j:

$$\frac{\Pr{}_{\boldsymbol{\theta};\boldsymbol{g}}(y_{i:j\succ k}=1\,|\,\boldsymbol{Y}=\boldsymbol{y}^{i:j\rightleftarrows k}\vee \boldsymbol{Y}=\boldsymbol{y})}{\Pr{}_{\boldsymbol{\theta};\boldsymbol{g}}(y_{i:j\succ k}=0\,|\,\boldsymbol{Y}=\boldsymbol{y}^{i:j\rightleftarrows k}\vee \boldsymbol{Y}=\boldsymbol{y})}=\frac{\Pr{}_{\boldsymbol{\theta};\boldsymbol{g}}(\boldsymbol{Y}=\boldsymbol{y}^{i:j\rightleftarrows k})}{\Pr{}_{\boldsymbol{\theta};\boldsymbol{g}}(\boldsymbol{Y}=\boldsymbol{y})}=\exp\{\boldsymbol{\theta}\cdot \Delta_{i,j}^{\diagup}\boldsymbol{g}(\boldsymbol{y})\}.$$

$$(5)$$

In this, the promotion statistic also reflects the conditional dependence structure of the model: if its form for a particular $\boldsymbol{g}(\cdot;\cdot)$ does not depend on a particular datum, using it in the model cannot induce conditional dependence on that datum. Thus, although we do not derive our model terms from a conditional dependence structure (cf. Robins et al. 1999), we can use them to examine the conditional dependence structure of the model for each term we consider.

Note that promotion statistics are mainly useful for complete orderings: if the ordering is partial, it is possible for i to promote j without demoting j^{+}. (See Butts 2007a for a parallel case involving models for one-to-one versus many-to-one assignments.)

3.2. *Terms for Exogenous Covariates*

We begin our quorum of substantively useful statistics by considering "exogenous" factors: those factors that would influence rankings by an ego i in a manner that is independent (in the probabilistic sense) of the

rankings of all other egos $i' \in N\backslash\{i\}$. (Indeed, none of the promotion statistics in this section depend on any other rankings.) Substantively, these factors are exogenous to the ranking process in that they are not, at least on the time scale of the process, mutable, or in that they operate independently of an ego i being able to observe or infer rankings or other salient states or actions (that are endogenous to the model) of any other ego i'.

3.2.1. *Attractiveness/Popularity Effects.*

For assessments of attractiveness, liking, and status, it is likely that egos' rankings will be influenced by some relatively stable (and exogenous) tendencies of particular alters to be rated more highly than others. For instance, assessments of physical attractiveness tend to be broadly consistent within a given cultural context, and such assessments correlate positively with physical attributes and performance characteristics (e.g., subtleties of dress and speech) that are usually difficult to alter over short time scales (Morse et al. 1974; Webster and Driskell 1983). Thus, we have the emic notion that some persons "are attractive," with the attribution regarded as a fixed trait of the person being assessed; while the reality is less trivial, stable factors governing attractiveness are sufficiently important that we may wish to capture them where possible. In other settings, institutionalized status characteristics (e.g., group membership, formal social roles) or the like may have similar effects (Berger, Cohen, and Zelditch 1972; Berger et al. 1977).

Regardless of source, we can treat these effects directly by positing some covariate vector $x \in \mathbb{R}^n$, associated with a statistic of the general form

$$g_A(y; x) = \sum_{(i,j,k) \in N^{3\neq}} y_{i:j \succ k}(x_j - x_k).$$

This statistic simply indexes the tendency for those with higher values on x to be ranked more highly than those with lower values. The promotion statistic associated with the above is

$$\Delta_{i,j}^{\nearrow} g_A(y; x) = 2(x_j - x_{j^+}),$$

that is, twice the difference between the attractiveness of j and the actor over whom j may be promoted. Therefore, the multiplicative effect of

this term on the odds (equation 5) of j's being ranked over j^+, conditional on the other rankings, is $\exp\left(2\theta_A(x_j - x_{j^+})\right)$.

   This coefficient of 2 appears in many of the promotion statistics proposed, because every promotion of j over j^+ affects two pairwise comparisons: $y_{i:j \succ j^+}$ and $y_{i:j^+ \succ j}$. It is tempting to eliminate it by redefining these statistics, multiplying them by half to compensate for this "redundancy." However that would muddle interpretation of these statistics applied to partially or weakly ordered data, because for those, it is possible for a change to affect $y_{i:j \succ k}$ without affecting $y_{i:k \succ j}$ (by creating or breaking ties); and even within the completely ordered domain, promotion statistics of more complex terms such as the local nonconformity introduced in Section 3.3.2 might not have this coefficient. Thus, in interpreting the parameter estimates of these models and the magnitudes of their effects, we must consider carefully the form of the corresponding promotion statistics.

   As expressed, g_A treats x as at least an interval scale; in other cases, the subtraction operator would need to be replaced with a more appropriate function. Although x may be an observed covariate, it is worth noting that this quantity is also a natural candidate for treatment via a random popularity effect (van Duijn, Snijders, and Zijlstra 2004).

3.2.2. *Difference/Similarity Effects.* Just as we may posit a differential tendency to "win" ranking contests overall, we may also posit that each actor i has exogenous characteristic $x_i \in \mathbb{X}$ such that alters "close to" or "far from" ego will be more likely to be highly ranked than those with the reverse attributes. (These assumptions are the basis of models such as spatial voting theory; Enelow and Hinich 1984.) This is a familiar application of homophily/heterophily to the rank-order case, and the implementation is straightforward:

$$g_H(y; x) = \sum_{(i,j,k) \in N^{3 \neq}} y_{i:j \succ k}\left[z(x_i, x_j) - z(x_i, x_k)\right], \qquad (6)$$

where $z : \mathbb{X}^2 \to \mathbb{R}$ is any function that is monotone increasing in the difference between its arguments. Thus, where this statistic is enhanced we expect "far" actors to outrank "near" ones (from the point of actor i), with the reverse holding where this statistic is suppressed. The atomic effect of this term is simply

$$\Delta_{i,j}^{\nearrow} g_H(y; x) = 2\left[z(x_i, x_j) - z(x_i, x_{j^+})\right].$$

As with attractiveness, difference effects can be based either on observed covariates or on latent quantities.

3.2.3. *Dyadic Covariates.* We can extend the above logic to general dyadic covariates. For instance, we may consider a case in which a within-context ranking is made by actors having ongoing social relationships; we might expect, then, that actors engaged in positive long-term relationships would tend to give preference to their partners within the specific rating context. Statistics for this behavior can be produced in this way:

$$g_{\text{Dyad}}(\boldsymbol{y};\boldsymbol{x}) = \sum_{(i,j,k)\in N^{3\neq}} y_{i:j\succ k}\left(x_{i,j} - x_{i,k}\right)$$

$$\Delta_{i,j}^{\nearrow}g_{\text{Dyad}}(\boldsymbol{y};\boldsymbol{x}) = 2\left(x_{i,j} - x_{i,j^+}\right).$$

Of course, the cases of attractiveness and difference described above are simply special cases of dyadic covariates, with particular structure imposed. (Notably, the matrix permutation family of Butts 2007b has a somewhat similar structure.)

3.2.4. *Comparison Covariates.* Finally, in the framework of pairwise comparison, the most general exogenous covariate form assigns a weight to each pairwise comparison by each ego:

$$g_{\text{P}}(\boldsymbol{y};\boldsymbol{x}) = \sum_{(i,j,k)\in N^{3\neq}} y_{i:j\succ k}\, x_{i,j,k},$$

for some $\boldsymbol{x} \in \mathbb{R}^{N^{3\neq}}$—assigning to each distinct ordered triple (i,j,k) a covariate value $x_{i,j,k}$—resulting in

$$\Delta_{i,j}^{\nearrow}g_{\text{P}}(\boldsymbol{y};\boldsymbol{x}) = \left(x_{i,j,j^+} - x_{i,j^+,j}\right).$$

This statistic has all other exogenous statistics as special cases.

3.3. *Terms for Endogenous Mechanisms*

We now turn to factors that are endogenous in the sense that, unlike exogenous factors, their effect on the rankings by Ego i does depend on rankings by other egos $i' \in N\backslash\{i\}$. Substantively, these are factors for phenomena that may plausibly arise in cases for which an ego observes or is able to infer the rankings of others.

3.3.1. *Global Conformity.* In many settings in which an ego is able to observe or infer the rankings of others, there is reason to presume that this will influence ego, so that he or she will tend to bring his or her own rankings into conformance with the rankings of others. This is certainly true in dominance or status rankings, for which there is considerable evidence that individuals can and do infer status ordering from observation of third-party judgments (e.g., see Anderson et al. 2006); this synchronization may even be explicit, as in certain types of gossip (wherein two or more parties "compare notes" on the relative status of their peers) (Dunbar 1997). The status of influence for relations such as relative liking is less clear but still plausible: an ego may take an alter's evaluations of the relative merits of other alters into account in assessing his or her own preferences, just as one can be influenced in one's judgment of the merits of food, art, or other experiential goods by the evaluations of others (Bordieu 1968). Finally, the mutual observability of rankings may produce in some settings a form of "conformity pressure" (e.g., Asch 1951), such that those displaying deviant rankings anticipate (and are possibly exposed to) sanction. The importance of influence processes in such settings is well documented.

To formalize influence in the ranking context, we must note that four elements are involved: an ego's assessment of two alters (say, j and k), and the assessment of those same alters by a distinct third party (say, $l \notin \{i, j, k\}$). Denoting ego by i, we note that ego's assessment of j and k is in conformity with l's assessment of j and k when $y_{i:j\succ k} = y_{l:j\succ k}$ and $y_{i:k\succ j} = y_{l:k\succ j}$. A natural statistic to summarize the degree of ratings nonconformity, then, is

$$g_{\text{GNC}}(\boldsymbol{y}) = \sum_{(i,j,k,l)\in N^{4\neq}} y_{l:j\succ k}\left(1 - y_{i:j\succ k}\right). \tag{7}$$

The promotion statistic for nonconformity can be derived by observing that when i promotes j over j^+, the statistic is incremented by 2 every other ego l who has j^+ ranked over j and decremented by 2 for every l who has j ranked over j^+ (1 for i conforming/disconforming to l and 1 for l conforming/disconforming to i). Thus,

$$\Delta_{i,j}^{\nearrow} g_{\text{GNC}}(\boldsymbol{y}) = 2 \sum_{l\in N\setminus\{i,j,j^+\}} (y_{l:j^+\succ j} - y_{l:j\succ j^+}), \tag{8}$$

a "vote" among the ls as to the relative ranking of j and j^+. Insofar as influence is active, g_{GNC} should be suppressed (and the associated parameter negative). In the typical case of total ordering within subjects, sufficiently strong suppression of g_{GNC} will force the population to converge to a universal consensus ranking; if the suppression is weaker, a looser but analogous set of states will be favored.

Unlike the promotion statistics of the terms for exogenous mechanisms, this one is a summation. This is unsurprising, because it reflects the dependence among the egos that it induces: whereas before promotion statistics did not depend on y at all, $\Delta_{i,j}^{\nearrow} g_{GNC}(y)$ depends specifically on how each of the other egos ranked the two specific alters j and j^+. Its form is also likely to affect its magnitude: whereas exogenous promotion statistics' magnitudes depend only on those of the covariates, the magnitude of $\Delta_{i,j}^{\nearrow} g_{GNC}(y)$ is likely to grow with network size, so care must be taken in interpreting how a particular value for the corresponding parameter estimate translates to the strength of the social force it models.

It should be noted that this form of influence and the random attractiveness effect, mentioned in Section 3.2.1, can both explain the same network feature: both heterogeneity in attractiveness and social influence induce an agreement in rankings, and the latter may be considered a marginal representation of the former, in a manner similar to that of a within-group correlation as a marginal reflection of a (conditional) random effects linear model.

3.3.2. *Local Conformity.* For global nonconformity g_{GNC}, the promotion statistic (equation 8) implies that i weights agreement with every other l equally, regardless of how i had ranked l. In some cases, it may be plausible that the salience of l for i may depend upon i's ranking of him or her; for instance, i may be more likely to attend to (and to conform to) those whom he or she ranks highly than those whom he or she ranks lower. A possible formalization of this is the notion that i's ranking of j would be influenced by l only if i ranks l over j, so only actors ranked above j influence i's rankings involving j. As with global conformity, we define this statistic, the *local nonconformity*, negatively: counting the number of instances in which an ego had ranked l over two alters j and k but then did not conform to l's ranking of j relative to k:

$$g_{LNC}(y) = \sum_{(i,j,k,l) \in N^{4\neq}} y_{i:\, l \succ j} y_{i:\, l \succ k} y_{l:\, j \succ k} (1 - y_{i:\, j \succ k}). \qquad (9)$$

The atomic effects for this statistic are somewhat complex:

$$\Delta^{\nearrow}_{i,j}g_{\mathrm{LNC}}(\boldsymbol{y}) = \sum_{k \in N \setminus \{i,j,j^+\}} (y_{i:\,k \succ j^+}y_{k:\,j^+ \succ j} - y_{i:\,k \succ j^+}y_{k:\,j \succ j^+} \tag{10a}$$

$$+ y_{k:\,i \succ j^+}y_{k:\,j^+ \succ j} - y_{k:\,i \succ j}y_{k:\,j \succ j^+} \tag{10b}$$

$$+ y_{j:\,k \succ j^+}y_{i:\,j^+ \succ k} - y_{j^+:\,k \succ j}y_{i:\,j \succ k}). \tag{10c}$$

They have, however, a meaningful interpretation. The two terms in line 10a represent the effect of i's bringing his or her ordering of j and j^+ into conformance (or disconformance) with those of some actor k whom i had ranked higher than them. The pair in line 10b represents the situation in which some actor k ranks i over j and j^+, so i promoting j over j^+ either creates or eliminates disconformance on the part of k. The pair in line 10c represents the notion that nonconformity is also created if actors i and j disagree on the ordering of j^+ and some actor k, so i promoting j over j^+ creates disconformance by making j's ordering of j^+ salient to i. (Ego i can resolve this tension either by changing the ranking of j^+ and k to conform with j [affecting equation 10a] or by demoting j to make his or her ranking less salient.)

This promotion statistic is also unusual among the others in that it does not have the coefficient of 2 first discussed in Section 3.2.1. In this case, it serves to underscore that the indicators $y_{i:\,j \succ k}$ and $y_{i:\,k \succ j}$ being coupled because of the complete ordering constraint does not imply that the act of ranking j over k has equivalent significance in the social process to the act of ranking k over j.

3.3.3. *Deference Aversion.* Influence, as defined above, deals with the mutual adjustment among raters regarding their relative assessments of third parties. When ego is a party to the rating in question, the situation becomes more complex. By assumption, ego does not explicitly self-rate; thus, ego cannot adjust toward alter's impression of him or her. In many settings, however, another mechanism may be active that will make alter's ranking of ego salient for ego's ranking of alter. In particular, consider the case in which higher rankings are associated with positive evaluation, such that being ranked below others is aversive. Moreover, let us assume that ego infers his or her own status via an implicit transitivity mechanism, such that if alter l ranks j above ego (i) and ego ranks l above j, then ego is for social purposes ranking

himself or herself below *l*. Under such circumstances, deference aversion may lead ego to resist ranking *l* above *j*.

To capture this notion with a statistic, we propose the following:

$$g_D(\boldsymbol{y}) = \sum_{(i,j,l)\in N^{3\neq}} y_{l:j\succ i} y_{i:l\succ j}. \tag{11}$$

We expect this statistic to be suppressed when deference aversion is present. The promotion statistic is incremented if j^+ had ranked *i* over *j*, because it creates a deference of j^+ to *i* via *j*; or if *j* had ranked j^+ over *i* because it creates a deference of *i* to *j* via *k*; and it is decremented if j^+ had ranked *j* over *i*, as it would eliminate the deference of *i* to j^+ via *j*; or if *j* had ranked *i* over j^+, as it would eliminate the deference of *j* to *i* via j^+. Thus,

$$\Delta_{i,j}^{\nearrow} g_D(\boldsymbol{y}) = 2(y_{j^+:i\succ j} + y_{j:j^+\succ i} - 1).$$

It is interesting to note that the principal effect of suppressing this statistic is actually to bring ego's rankings in line with those of alter, somewhat akin to the reciprocity or mutuality in binary relations. Specifically, if there are *r* persons ranked by alter as being above ego, then ego will also tend to rank those same *r* persons as being above alter. Where a total order is present, ego and alter will thus tend to give each other the same rank (and, indeed, to agree on those persons having higher ranks). Of course, applying this logic to all pairs suggests pressure toward equality, which is impossible to achieve in the total order case (but not necessarily for others). Even in the case of total orders, however, considerable variation in g_D is possible, with lower values indicating rating structures in which agreement between raters on high-ranked alters is maximized.

3.4. *Consistency across Settings*

When ranking the same alters among multiple settings—across time or across rubrics—there is reason to expect that ego will tend to exhibit consistency in alter ratings. Across time, this is an exogenous effect, because earlier rankings cannot be influenced by later rankings. Across rubrics, it may be endogenous. Here we assume two rating structures, $\boldsymbol{y}$ and $\boldsymbol{y}'$, on vertex sets *N* and *N'*, such that some sets $N_s = N \cap N'$ of actors are involved in both networks. For convenience in notation, we take the

labeling of the members of N_s to be the same in both N and N'. Given this, our statistic measuring inconsistency is simply

$$g_{\mathrm{I}}(\boldsymbol{y};\boldsymbol{y}') = \sum_{(i,j,k)\in N_s^{3\neq}} \left[y_{i:j\succ k}(1 - y'_{i:j\succ k}) + (1 - y_{i:j\succ k})y'_{i:j\succ k} \right], \quad (12)$$

with the promotion statistic being simply

$$\Delta_{i,j}^{\nearrow} g_{\mathrm{I}}(\boldsymbol{y};\boldsymbol{y}') = 2(y'_{i:j^+\succ j} - y'_{i:j\succ j^+}).$$

Because g_{I} measures the discordant pairs of rankings in $\boldsymbol{y}$ versus $\boldsymbol{y}'$, suppressing it implies higher levels of cross-context consistency.

Statistic (12) treats all disagreements between $\boldsymbol{y}$ and $\boldsymbol{y}'$ as equivalent. It may be the case, however, that only some disagreements are of interest, or disagreements themselves need to be modeled. This can be facilitated by a more general form of g_{I}. Given weights $\boldsymbol{x} \in \mathbb{R}^{N^{3\neq}}$ (symmetric for complete orderings, such that $x_{i,j,k} \equiv x_{i,k,j}$), let the *weighted inconsistency* of $\boldsymbol{y}$ versus $\boldsymbol{y}'$ be defined as

$$g_{\mathrm{I}}(\boldsymbol{y};\boldsymbol{y}',\boldsymbol{x}) = \sum_{(i,j,k)\in N_s^{3\neq}} \left[y_{i:j\succ k}(1 - y'_{i:j\succ k}) + (1 - y_{i:j\succ k})y'_{i:j\succ k} \right] x_{i,j,k}. \quad (13)$$

Note that $\boldsymbol{x}$ can, itself, be parametrized to model factors affecting disagreement between two rankings, making it a potentially interesting basis for hierarchical modeling.

These statistics are used extensively in the example in Section 4.2, in which they are used to examine the accuracy of informants' self-reported interaction frequencies.

4. EXAMPLES

4.1. *Dynamics of the Acquaintance Process*

From 1953 to 1956, a research group led by Theodore Newcomb (1961) conducted an experimental study of acquaintance and friendship formation. In each of the two study years, 17 men attending the University of Michigan—all transfer students with no prior acquaintance among them—were recruited to live in off-campus fraternity-style housing. Demographic, attitudinal, and sociometric information was collected about the subjects. In particular, in the second year, at each of 15 weekly time points (with week 9 being missing), each participant was

asked to rate all other participants on "favorableness of feeling," with ratings forced to be distinct and converted to ranks (pp. 32–34). These data represent an example of longitudinal data of complete ranks, and the data from the second year of the study in particular have been used to study the formation of interpersonal relationships by Newcomb (1956), Breiger et al. (1975), White et al. (1976), Arabie et al. (1978), Wasserman (1980), Pattison (1982), Nakao and Romney (1993), Doreian et al. (1996), Krackhardt and Handcock (2007), and many others.

We use ERGMs for rank-order data to study this network, examining the social forces relevant to its structure and its evolution over time. We take two distinct approaches: (1) cross-sectional, in which each time point's network structure is modeled on its own, and (2) dynamic, in which each time point but the first is effectively modeled as a change from the previous time point's rankings.

4.1.1. *Cross-sectional Analysis.* Demographic data, including age, religion, and political views of the subjects, were gathered. Also, within the house, the subjects were assigned to rooms, spread over two floors of the house—three one-occupant rooms, four two-occupant rooms, and two three-occupant rooms (Newcomb 1961:67–68). Furthermore, although some subjects were assigned to rooms at random, others were assigned with an aim to maximize (for some rooms) and minimize (for other rooms) the roommates' compatibility as understood by the researchers (pp. 216–20). If available, all of these factors could be used as predictors in our modeling framework via terms introduced in Section 3.2. Sadly, to our best knowledge, none of these elements of the Newcomb data survive, leaving us to focus on endogenous effects (although the "birds of a feather or friend of a friend" [Goodreau et al. 2008a] caveat applies). For each of the 15 networks, we model deference aversion—via $g_D(\boldsymbol{Y}^t)$ (equation 11)—and global—via $g_{GNC}(\boldsymbol{Y}^t)$ (equation 7)—and local—via $g_{LNC}(\boldsymbol{Y}^t)$ (equation 9)—conformity. Per Section 3.3.1, the suppressing of the global nonconformity statistic produces the same effect as latent actor attractiveness, so it may, in this case, be viewed as modeling latent heterogeneity in popularity.

We report the maximum likelihood estimates for each of the terms over time in Table 1 and plot them in Figure 4. Deference aversion is significant (the coefficient on the deference statistic is negative)

throughout the evolution of rankings, starting with the first point of observation. This is consistent with the finding of Newcomb (1956), Doreian et al. (1996), and others that "friendships are reciprocated immediately." Our analysis, however, suggests deepening deference aversion over time, not reaching its ultimate magnitude until week 4 or 7. (Informally, the estimated Kendall's rank correlation between the parameter estimate and week number is $\hat{\tau} = -0.41$, significant with p value $= .036$.) One explanation for this difference is that prior analyses, including that of Doreian et al. (1996), dichotomized the dyads in the network. Our approach uses the entire ranking and may thus be more precise.

The local nonconformity term is also significant for all but the first observation point, although its effects seem to emerge more gradually than those of deference aversion (Kendall's $\hat{\tau} = -0.64$, p value $<.001$), leveling off around week 7 or perhaps even later. This does agree with Doreian et al. (1996), although others have suggested earlier and later times when the network stabilized.

In the presence of the local nonconformity term, the global nonconformity term is not significant: there does not appear to be a significant overall consensus in ratings, although Newcomb (1956) reported that three specific subjects were generally disliked by everyone, including one another, so perhaps failing to detect this is a result of lack of power and presence of the local nonconformity term. Notably, the estimated correlation between the local and global nonconformity parameters (within each week's fit) is strongly negative and consistent (-0.87 $--0.77$). This suggests that these terms explain similar behavior, though significance of one and not the other indicates that conformity is primarily to those more liked.

4.1.2. *Dynamic Analysis.* We now turn to modeling the evolution of the rankings over time. For our dynamic analysis, we use a simple Markov formulation similar to those of Krackhardt and Handcock (2007) and Hanneke, Fu, and Xing (2010):

$$\text{Pr}_{\boldsymbol{\theta};\boldsymbol{g}}(\boldsymbol{Y}^t = \boldsymbol{y}^t \,|\, \boldsymbol{Y}^{t-1} = \boldsymbol{y}^{t-1}) = \frac{\exp\{\boldsymbol{\theta} \cdot \boldsymbol{g}(\boldsymbol{y}^t; \boldsymbol{y}^{t-1})\}}{\kappa_{\boldsymbol{g}}(\boldsymbol{\theta}, \boldsymbol{y}^{t-1})},$$

having the normalizing constant

Table 1. Results for Cross-sectional Analysis of Newcomb's Data

		Estimates (Standard Errors)	
		Nonconformity	
Week	Deference	Global	Local
0	−.153 (.039)***	−.004 (.003)	−.010 (.010)
1	−.218 (.047)***	−.001 (.003)	−.020 (.009)*
2	−.221 (.047)***	.002 (.003)	−.031 (.009)***
3	−.209 (.046)***	.004 (.003)	−.036 (.008)***
4	−.288 (.060)***	.001 (.003)	−.034 (.008)***
5	−.251 (.058)***	.001 (.003)	−.040 (.008)***
6	−.236 (.057)***	.000 (.003)	−.037 (.008)***
7	−.399 (.081)***	.003 (.002)	−.045 (.007)***
8	−.373 (.073)***	.001 (.003)	−.037 (.007)***
9	−.312 (.070)***	.003 (.003)	−.046 (.007)***
10	−.254 (.060)***	.003 (.003)	−.045 (.008)***
11	−.299 (.066)***	−.000 (.003)	−.036 (.007)***
12	−.174 (.050)***	.002 (.003)	−.047 (.008)***
13	−.365 (.078)***	.002 (.003)	−.045 (.009)***
14	−.337 (.076)***	.001 (.003)	−.042 (.008)***

Note: Significance levels: $.05 \geq * > .01 \geq ** > .001 \geq ***$.

$$\kappa_g(\boldsymbol{\theta}, \boldsymbol{y}^{t-1}) = \sum_{\boldsymbol{y}' \in \mathcal{Y}} \exp\{\boldsymbol{\theta} \cdot \boldsymbol{g}(\boldsymbol{y}'; \boldsymbol{y}^{t-1})\}.$$

For each of the 14 *transitions* between successive networks, in addition to the three terms used in the cross-sectional analysis, we model inconsistency over time via $g_I(\boldsymbol{Y}^t; \boldsymbol{Y}^{t-1})$ (equation 12). This term effectively absorbs the structure in the network at t that is due to inertia and to social forces operating in the time periods prior to $t-1$, and thus the other terms model the social forces affecting only the *changes* in the rankings over time.

Because we seek to examine the strengths of the factors over time, we use time-varying parameters (although Krivitsky and Handcock [2014] showed that the approach of Hunter and Handcock [2006] can be applied to series of networks or transitions as well). Week 9 rankings were not reported. Because of this, for week 10, we fit the parameters for transition from week 8.

The maximum likelihood estimates for each transition are reported in Table 2 and visualized in Figure 5. The estimates for the transition from week 8 to week 10 do not appear to be qualitatively different from those
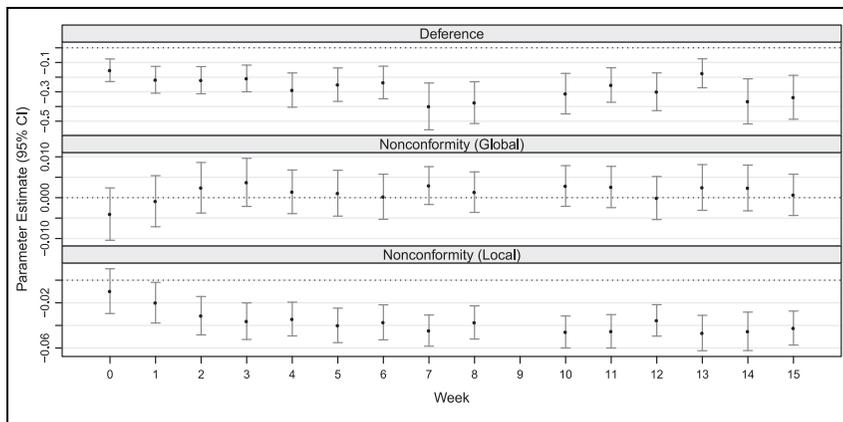
Figure 4. Estimated coefficients for the cross-sectional model fit to each week's rankings in Newcomb's fraternity. Error bars are at 95 percent confidence.

for nearby transitions. In particular, inconsistency does not appear to be higher over this particular two-week period.

The clear downward trend ($\hat{\tau} = -0.54$, *p* value = .007) in inconsistency over successive weeks suggests that the rankings are initially in flux as the acquaintance process takes place, solidifying over time. As before, global nonconformity is not a significant factor. The parameter estimates for the other two factors still appear to be, on the whole, significant, but they are uniformly smaller in magnitude compared with those of the corresponding weeks in the cross-sectional analysis and are less precisely estimated (as represented by uniformly greater standard errors). This is because they embody only the structure of *changes* in the network over the week, rather than embodying the structure of the whole network, and information to infer their strength is drawn only from those changes. This means that some "instant" social effects such as friendship reciprocation have been absorbed into the week 0 observation, which is not modeled in the dynamic analysis. Snijders (1996) reported similar conclusions.

In contrast to the cross-sectional analysis, neither deference nor local nonconformity appears to have a significant monotone trend over time (both correlations have *p* value $\geq$ .5). That is, although they are time varying when viewed cross-sectionally, the effects of these social forces over and above inertia are fairly consistent over time, at least after the initial time point. This suggests that this modeling approach may be

successfully isolating social forces as they affect actors' behavior over time from the effects of preexisting configurations.

4.2. *Informant Accuracy*

In the late 1970s, Bernard et al. (1984) conducted a series of studies to assess the accuracy of retrospective sociometric surveys of several types. In each study, respondents in a social network—deaf teletype users; amateur radio operators; office workers at a firm; students in a fraternity; and faculty members, graduate students, and staff members in an academic program—had their social interactions observed or recorded and were asked, in retrospect, to indicate others in their network with whom they interacted, allowing recalled and observed network structures to be compared. In the latter study, conducted in a graduate program in technology education at West Virginia University, the 34 subjects had the frequency of their interactions recorded by a team of observers over the course of a week, and then each subject was asked to provide a complete ranking of the other subjects on "most to least communication that week" (Bernard and Killworth 1977). This produced a complete ranking, suitable for analysis using our methods.

4.2.1. *Modeling Inconsistency.* In this application, we use models with sufficient statistics of the form of equations (12) and (13) to assess factors that appear to affect accuracy of rankings. Let $y \in \mathcal{Y}$ be the reported rankings; and let $y^{\text{ct.}} \in \mathbb{N}_0^{\mathbb{Y}}$ be a weighted symmetric graph of the observed frequencies of interaction, so that $y_{i,j}^{\text{ct.}}$ is the number of times i and j were observed interacting. Like y^{t-1} in the previous example, it is exogenous in our framework.

For notational convenience, we re-express equations (3) and (13) as

$$\Pr\nolimits_{\theta;g,x}(Y = y \mid Y^{\text{ct.}} = y^{\text{ct.}}) =$$

$$\frac{\exp\left[\sum_{(i,j,k)\in N^{3\neq}} \{y_{i:j\succ k}(1 - y_{i:j\succ k}^{\text{ct.}}) + (1 - y_{i:j\succ k})y_{i:j\succ k}^{\text{ct.}}\} w_{i,j,k}(\theta;x)\right]}{\kappa_{g,x}(\theta)}, \quad (14a)$$

where

$$w_{i,j,k}(\theta;x) = \theta \cdot x_{i,j,k}, \quad (14b)$$

for covariate $n \times n \times n \times p$-array $x \in \mathbb{R}^{N^{3\neq} \times \{1 \cdots p\}}$, so that $x_{i,j,k}$ is the p-vector of covariates associated with i's comparison of j and k, and, as

Table 2. Results for Dynamic Analysis of Newcomb's Data

| | Estimates (Standard Errors) | | Nonconformity | |
			Global	Local
Week Transition	Inconsistency with Prior Week	Deference		
0 → 1	−.135 (.015)***	−.192 (.048)***	.001 (.003)	−.019 (.009)*
1 → 2	−.212 (.020)***	−.165 (.051)**	.005 (.004)	−.032 (.010)**
2 → 3	−.244 (.021)***	−.130 (.055)*	.002 (.004)	−.030 (.011)**
3 → 4	−.281 (.026)***	−.206 (.071)**	−.007 (.004)	−.026 (.010)**
4 → 5	−.292 (.026)***	−.128 (.068)	−.005 (.004)	−.029 (.010)**
5 → 6	−.348 (.027)***	−.133 (.071)	−.003 (.005)	−.016 (.011)
6 → 7	−.389 (.030)***	−.392 (.091)***	.008 (.006)	−.046 (.011)***
7 → 8	−.309 (.028)***	−.218 (.083)**	−.002 (.005)	−.013 (.011)
8 → 10	−.301 (.026)***	−.157 (.077)*	−.000 (.005)	−.034 (.011)**
10 → 11	−.289 (.025)***	−.126 (.070)	−.001 (.004)	−.024 (.010)*
11 → 12	−.324 (.027)***	−.216 (.078)**	−.006 (.005)	−.009 (.011)
12 → 13	−.373 (.032)***	−.017 (.066)	.006 (.006)	−.042 (.012)***
13 → 14	−.345 (.029)***	−.343 (.090)***	−.003 (.005)	−.031 (.010)**
14 → 15	−.314 (.027)***	−.190 (.085)*	−.004 (.004)	−.025 (.011)*

Note: Significance levels: .05 ≥ * > .01 ≥ ** > .001 ≥ ***.
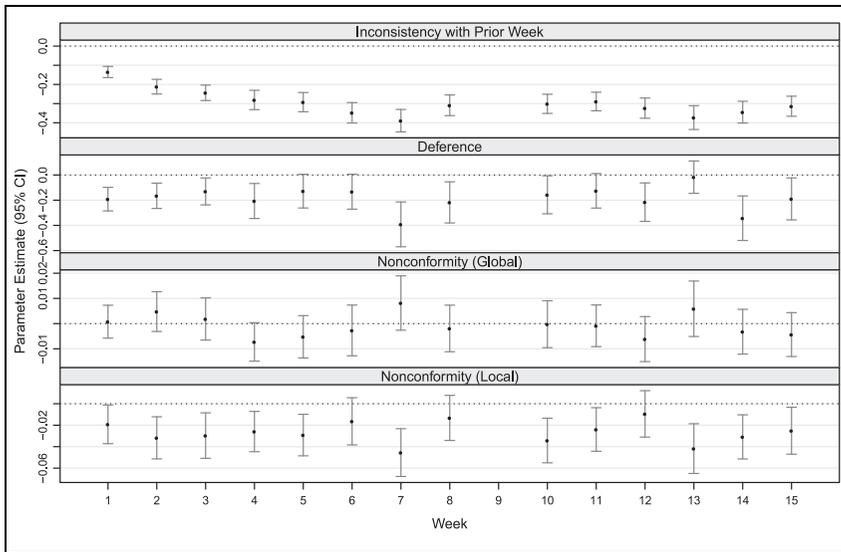
Figure 5. Estimated coefficients for the longitudinal model fit to each week's rankings in Newcomb's fraternity. Error bars are at 95 percent confidence.

before, $x_{i,j,k} \equiv x_{i,k,j}$. This allows us to model inconsistency in a form resembling logistic regression. Unlike logistic regression for accuracy of pairwise comparisons, this model takes into account the dependence between the comparisons that is induced by the structure of the sample space (i.e., that $y_{i:j \succ k} \land y_{i:k \succ l} \Rightarrow y_{i:j \succ l}$).

Because there are ties among the observed interaction frequencies (i.e., where $y_{i,j}^{ct.} = y_{i,k}^{ct.}$, so i interacted with j and k equally often), while the reported rankings are forced to be complete (no ties were allowed), there is no configuration of rankings $y \in \mathcal{Y}$, such that statistic (12) is 0—that the reported rankings are completely consistent with those observed. This reveals an interesting property of the proposed class of models: because the comparisons that are tied in the observed frequencies simply add a constant to their sufficient statistic, their effect on the likelihood is canceled by the normalizing constant. That is, the model and the estimation are affected only by those inconsistent comparisons that had the possibility of being consistent under the model in the first place.

For convenience, let $y_{i,j}^{obs.}$ be the *observed* rank of j among those with whom i had interacted, with 33 being code for the highest frequency, 1

being code for the lowest frequency, and ranks for tied alters (i.e., $y_{i,j}^{\text{ct.}} = y_{i,k}^{\text{ct.}}$, for $k \neq j$) being computed by averaging the ranks that these alters share.

4.2.2. *Effect of Frequency of Interaction.* The first question we address is whether the magnitude of the difference in the frequency of interaction affects the accuracy. That is, if i's frequency of interaction with j differs from i's frequency of interactions with k by more than i's frequency of interaction with j differs from i's frequency of interaction with l, is i more likely to rank j and k accurately than j and l?

To answer this, we begin by fitting a simple model with two covariates: $x_{i,j,k,1} = 1$ in the form of equation (14), equivalent to plain inconsistency (equation 12); and $x_{i,j,k,2} = |y_{i,j}^{\text{ct.}} - y_{i,k}^{\text{ct.}}|$, the absolute difference between the interaction frequency of i with j and k. We report the results in Table 3. Greater difference in interaction frequency of two alters does appear to lead to greater accuracy (i.e., lower inaccuracy) in reporting their relative ranks.

We also fit a similar model in which we replace frequency difference with frequency rank difference: $x_{i,j,k,2} = |y_{i,j}^{\text{obs.}} - y_{i,k}^{\text{obs.}}|$. We find that the effect of rank difference is, as expected, negative, but it lacks statistical significance. This lack of significance may be counterintuitive, but it is in fact a consequence of the constraints imposed by the sample space. Intuitively, given that i ranks j and k adjacently, an inaccurate reporting of the pairwise comparison of these alters (e.g., $y_{i:j \succ k}^{\text{ct.}}$ but $y_{i:k \succ j}$) does not entail misreporting any other pairwise comparisons, including those involving j or k; but if j and k are some $d > 1$ ranks apart in $\mathbf{y}^{\text{ct.}}$—they have $d - 1$ other alters between them—then inaccurately reporting the pairwise comparison of j and k entails inaccurately reporting the pairwise comparisons $y_{i:j \succ l}^{\text{ct.}}$ and/or $y_{i:k \succ l}^{\text{ct.}}$ for every l ranked between j and k. It can be shown easily that any configuration $\mathbf{y}$ in which $y_{i:k \succ j} \neq y_{i:j \succ k}^{\text{ct.}}$ must also misreport at least $d - 1$ such comparisons. Thus, even a model with no rank difference effect and only a baseline inconsistency effect would already heavily penalize inaccurate reporting of comparisons between distantly ranked alters.

4.2.3. *Effect of Salience.* The second question that we address is whether the accuracy of reported ranking is affected by the positions of those being ranked. Can an ego i better discern the ranking of those

Table 3. Effect of Frequency of Interaction on Reporting Inaccuracy

	Estimates (Standard Errors)	
Term	Frequency	Rank
Intercept	−.066 (.020)**	−.159 (.023)***
Frequency difference	−.018 (.003)***	
Frequency rank difference		−.003 (.003)

Note: Significance levels: .05 $\geq$ * > .01 $\geq$ ** > .001 $\geq$ ***.

with whom he or she interacts the most? Is he or she more accurate at the extremes?

To answer this, we fit a model for inconsistency that is a quadratic polynomial in rank values. More concretely, in the form of equation (14), $w_{i,j,k}(\boldsymbol{\theta};\boldsymbol{x})$ has the covariate vector

$$\boldsymbol{x}_{i,j,k} = \left[1, y_{i,j}^{\text{obs.}} + y_{i,k}^{\text{obs.}}, (y_{i,j}^{\text{obs.}})^2 + (y_{i,k}^{\text{obs.}})^2, y_{i,j}^{\text{obs.}} y_{i,k}^{\text{obs.}} \right], \qquad (15)$$

inducing a model in which the inconsistency of reported with observed is modeled as a second-degree polynomial function of the observed ranks of the alters being compared; and this function is symmetric for these two alters. The statistics in equation (15) represent baseline inconsistency, the linear effect of the ranks of the alters being compared, their quadratic effect, and their interaction effect, respectively.

In fitting this model, we found that it suffers from collinearity, which impedes inference; so to improve its numeric conditioning, we fit an equivalent model, using rescaled and centered quantiles, evaluating $y_{i,j}^{\text{q.}} \equiv (y_{i,j}^{\text{obs.}} - 17)/32$ and substituting $\boldsymbol{y}^{\text{q.}}$ for $\boldsymbol{y}^{\text{obs.}}$ in equation (15).

We report results from this fit in Table 4. All four covariates appear to be highly significant. Somewhat surprisingly, the higher ranked alters appear to have slightly higher inconsistency between observed and reported. However, the negative coefficient on the quadratic term suggests that the middle ranks are reported with the least accuracy of all. We show the predicted inconsistency weight $w_{i,j,k}(\hat{\boldsymbol{\theta}};\mathbf{x})$ as a function of their observed ranks in Figure 6. From this, it appears that, indeed, for alters whose observed rankings are close together, the accuracy is lowest if the alters are ranked in the middle. (For alters whose observed rankings are far apart, the accuracy is higher.)

Table 4. The Effects of Alter Positions on Reporting Inaccuracy

Term	Estimate (Standard Error)
Intercept	$-.068(.024)$**
Linear	$.107(.018)$***
Quadratic	$-1.300(.296)$***
Interaction	$1.765(.512)$***

Note: Significance levels: $.05 \geq * > .01 \geq ** > .001 \geq ***$.

5. DISCUSSION

As befits the long history of ordinal data analysis in the social sciences, many extensions and applications of the present framework are possible. Here, we briefly discuss two such directions: (1) the extension of the current framework to "bipartite" rank data and (2) the use of our framework in settings that imply particular kinds of ordinal constraints (e.g., partial orders, semi-orders, and incompletely observed total orders).

5.1. *Extension to "Bipartite" Rank Data*

Although our focus here has been on the classic sociometric setting in which a group of individuals is asked to rank each other with respect to some dimension (e.g., liking), our modeling framework can easily accommodate other cases as well. One such setting is the case of "bipartite" rank data, in which members of a given group are asked to rank a set of objects not including the group members themselves. Examples of such data include preference rankings of political candidates, organizations, or policy positions; ordinal judgments regarding physical objects or perceptual stimuli; liking or other rankings of non-group members, and so on. Although such data are widely analyzed throughout the social sciences using traditional techniques, the contribution of our approach is the ability to model *interdependence among raters* in a natural way. For instance, the global conformity statistic of Section 3.3.1 can be used to capture a general tendency of group members to converge on a common rating of a set of objects; likewise, the consistency effects of Section 3.4 have the same meaning in the "bipartite" setting as in a standard sociometric setting. The exogenous effects of Section 3.2.1 can be used to capture differences in the net attractiveness of objects to members of the rating group, and dyadic
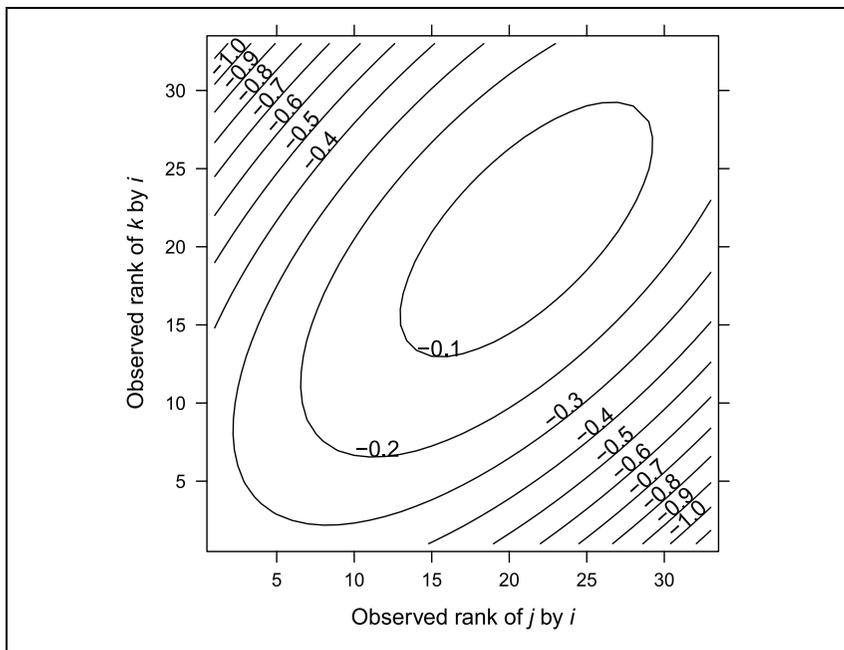
Figure 6. Predicted effect of alter positions on reporting inaccuracy.

covariates (Section 3.2.3) can be used to measure effects related to the tendency of particular raters or subgroups of raters to give higher/lower ratings to particular objects. On the other hand, statistics that depend upon the ratings *by* the object (e.g., local conformity) are clearly meaningless in a bipartite context, and should not be used.

The bipartite case suggests certain statistics that may prove especially useful for assessing influence in structured groups. For instance, consider the case in which our set of raters, N_e, interacts via a known, fixed social network with adjacency matrix x. Let N_a be the set of objects to be ranked. We may then capture the tendency for those adjacent in x to rate objects in N_a (dis)similarly via the *dyadic nonconformity statistic*,

$$g_{DNC}(y;x) = \sum_{(i,l) \in N_e^{2\neq}} x_{i,l} \sum_{(j,k) \in N_a^{2\neq}} y_{l:j\succ k}\left(1 - y_{i:j\succ k}\right).$$

To the extent that adjacent actors influence each other to form similar views of the object set, g_{DNC} will be suppressed; the associated parameter is hence analogous (up to a sign change) to the autocorrelation

parameter in a standard linear network autocorrelation model (LNAM) (Cliff and Ord 1973; Doreian 1990). As with the LNAM, the adjacency structure (analogous to the weight matrix) need not be dichotomous, and it can contain continuous measures of exposure, proximity, similarity, group comembership, or the like. Also like the LNAM, the adjacency/weight matrix is taken to be exogenous and fixed; models in which object ratings and interpersonal relationships coevolve would be a promising direction for future research in this area.

Simulation and estimation for bipartite rank data requires modifying the support of the associated model to include only the observable ranks, and eliminating impermissible rating triads from the proposals in Algorithm A1 in Appendix A.2. These are straightforward changes to the base implementation and are not discussed in detail here.

5.2. *Considerations Relating to Types of Orderings*

It should be noted that in our development, we focus on the case of orderings that are defined from the psychological process in question, rather than data that are ordinal simply due to limitations in measurement (such as count or continuous data observed only as ranks). Although the present framework may in some cases be useful for such data (e.g., to avoid having to model the count distribution), the assumptions involved—for example, in the choice of sufficient statistics (and their interpretation)—may be quite different.

Although we have focused on complete orderings, the above distinction gains further importance when considering partial orderings: the case of partial orderings being a property of underlying psychological phenomena is substantively different from the case of incomplete orderings arising from measurement itself. A well-known example of the latter is a frequently used sociometric survey design that asks each ego to rank his or her top k alters with respect to some criterion (liking, interaction frequency, etc.). Cases of unobserved ranking are better handled by means of a latent variable framework in which a probability distribution is placed on the complete data, and in which likelihood is assessed via the marginalization of the complete data conditional on that observed. (Plackett-Luce models for top-k ranking data [Plackett 1975] do this implicitly.) Such a strategy has been used in the traditional ERGM framework by Handcock and Gile (2010), and can be employed here as well.

In contrast, orderings in which the alters may be *substantively* tied or incomparable with one another pose different challenges. A close examination of Sampson's (1968) data reveals that some of the novitiates were in fact recorded assigning equal ranks to some of those they nominated, and these would be substantively tied in an underlying partial ordering. Tied ratings in data like those of Johnson et al. (2003) could be interpreted either way: either that the scale of 0 to 10 was not sufficiently granular to encode small differences in degrees of interaction, meaning that which one of the tied alters was actually higher is unobserved, or that an ego's choice to rate two alters equally means that their degrees of interaction were substantively the same.

In the complete ordering case, there exists a distribution of rank orderings that is unambiguously uniform and can thus serve as the baseline distribution (*reference measure*) for the exponential family (e.g., see Barndorff-Nielsen 1978:115–16). In the partially ordered case, there is no natural baseline that is unambiguously uniform. For example, an ego may rank all $n - 1$ alters equally (with only one such "ranking" possible) or he or she may assign a distinct rank to each (with $(n - 1)!$ possible rankings), or anything in between. Modeling of partial orderings, therefore, requires modeling not only the actors' ranking propensities but also their choices whether to rank or not to rank, and, in particular, the baseline distribution of these choices, when $\boldsymbol{\theta} = \mathbf{0}$. How to capture such behavior in a cognitively realistic way is not currently known.

Despite our focus on complete orderings, our techniques can (with appropriate choice of reference measure) be generalized to the weaker case: all of the model statistics that we present in Section 3 can be applied to partially ordered network data without modification, because they only make references to the network of interest through the indicator $y_{:,:\succ:}$.

6. CONCLUSION

Rank-order data are a cornerstone of sociometric measurement, but principled treatment of such data in an interpersonal context poses significant statistical challenges. Here, we have shown how statistical exponential families may be used to generalize the now well-known ERGM framework to the rank-order case. We have also introduced a corresponding set of sufficient statistics that are appropriate for use when only within-ego ordinal judgments are psychologically

meaningful, a restriction that is important when modeling such data. As with conventional ERGMs, a wide range of statistics may be posited to capture alternative psychological and social mechanisms; the ability to evaluate and compare competing models on the basis of such distinct alternatives is one of the strengths of the statistical approach.

In our assumption that the network must be modeled through $y_{\cdot:\cdot\succ\cdot}$, we have discarded any information associated with rank as such. Where measurements are *truly* ordinal, this is the appropriate choice; nevertheless, it has been noted (e.g., by Levine 1993) that putatively ordinal data can in practice contain more information than a strict ordinal interpretation would allow. As we show in our example in Section 4.2, rank values and rank differences can be incorporated into the models as well, though that example only uses them exogenously. Our framework does not, fundamentally, preclude incorporating such effects, and, in fact, it permits separating and testing their significance over and above the purely comparison-based effects. This is also a subject for ongoing work.

Finally, we note that development of ERGMs for rank-order data opens the way to a rich family of novel statistical models for phenomena such as interdependent choice behavior in group context and social influence on preferences. Particularly because of their suitability for data collected in observational settings, rank-order ERGMs provide a useful tool for considering both new and classic problems in social psychology and the study of decision making.

APPENDIX A

Inference and Implementation

The proposed class of models is a finite exponential family over a finite sample space. Its natural parameter space $\boldsymbol{\Theta}_N = \{\boldsymbol{\theta}' : \kappa_g(\boldsymbol{\theta}') < \infty\} = \mathbb{R}^p$, is an open set, so the models from this family are regular and possess all of the inferential properties of such families (Brown 1986:1–2). Here, we briefly discuss practical issues associated with fitting the models of this class.

A.1. *Implementation.* The sample space of a complete ranking ERGM is finite but large ($|\mathcal{Y}| = \{(n-1)!\}^n$), so even for networks of modest size, evaluating the normalizing constant (equation 4) is not computationally feasible. However, provided a method for sampling from this ERGM distribution $\Pr_{\boldsymbol{\theta}:g}(\cdot)$ exists, the Markov chain Monte Carlo maximum likelihood estimation

(MCMC MLE) method can be used to fit the model (Geyer and Thompson 1992; Hunter and Handcock 2006; Krivitsky 2012). We review this technique and describe a simple algorithm for sampling from a complete ordering ERGM in Section A.2. Bayesian inference using exchange algorithms is also possible, but more computationally expensive (Caimo and Friel 2011).

We base our implementation on the valued ERGM extensions of Krivitsky (2012) to the R (R Core Team 2014) package ERGM (Hunter, Handcock, et al. 2008; Handcock et al. 2014), and we have publicly released our extensions in a new package, ergm.rank.

Lacking a general procedure such as maximum pseudo-likelihood estimation (MPLE) (Strauss and Ikeda 1990) to use as a starting point for optimization, we initiated the MCMC MLE optimization at $\theta = 0$ in each of our applications. This parameter configuration corresponds to a uniform distribution on the space of possible rankings, and it is therefore the safest configuration to start with. Although a poor starting value can cause MCMC MLE to fail, techniques such as the stepping algorithm of Hummel, Hunter, and Handcock (2012) can be used to ameliorate its effects, and a form of it is used here as well. To confirm the effectiveness of these techniques even with a poor starting value, we performed a small parameter recovery simulation study, which we report in the Online Supplement, along with MCMC diagnostics for all of the fits.

Given the current wave of interest in "large" networks, computationally scalable estimation has been an increasing focus of research in the social network community. Relative to population-scale networks, the networks considered here have been fairly small (i.e., tens of vertices), and our focus has been on developing methods that work well on this latter scale. This focus is motivated by the fact that sociometric data of the form treated here are typically of interest only in group or organizational settings in which all members of the network are salient and well-known to each other; because such networks are by nature fairly small (e.g., rarely larger than 20 to 50 persons), highly scalable techniques are less critical for rank-order models than for binary networks. That said, computationally scalable estimation (particularly for the above-described top-k design) is an interesting challenge for future research in this area.

A.2. *An Overview of Monte Carlo MLE and a Simple Sampling Algorithm for a Complete Ranking ERGM.* In this appendix, we review the basic algorithm of Geyer and Thompson (1992), its application to ERGMs for dichotomous data by Hunter and Handcock (2006), and its extension to valued ERGMs by Krivitsky (2012), and we outline the MCMC sampling algorithm that it requires for rank data.

The challenge in fitting the model specified by equation (3) is that the likelihood contains an intractable normalizing constant (equation 4), which is a summation over a large, finite sample space of cardinality $\{(n-1)!\}^n$. This summation can be approximated using importance sampling Monte Carlo integration. In practice, this is highly inefficient. For the purposes of likelihood maximization, however, it suffices to approximate the ratio of the normalizing constants $\kappa_g(\boldsymbol{\theta}')/\kappa_g(\boldsymbol{\theta})$ for some two parameter configurations $\boldsymbol{\theta}'$ and $\boldsymbol{\theta}$: maximizing

$$\frac{\Pr_{\boldsymbol{\theta}';g}(Y=y)}{\Pr_{\boldsymbol{\theta};g}(Y=y)} = \frac{\exp\{\boldsymbol{\theta}'\cdot g(y)\}}{\kappa_g(\boldsymbol{\theta}')} \Big/ \frac{\exp\{\boldsymbol{\theta}\cdot g(y)\}}{\kappa_g(\boldsymbol{\theta})} = \frac{\exp\{\boldsymbol{\theta}'\cdot g(y)\}}{\exp\{\boldsymbol{\theta}\cdot g(y)\}} \Big/ \frac{\kappa_g(\boldsymbol{\theta}')}{\kappa_g(\boldsymbol{\theta})} \tag{A1}$$

with respect to $\boldsymbol{\theta}'$ will produce the MLE as well as maximizing $\Pr_{\boldsymbol{\theta}';g}(Y=y)$ itself would.

In estimating this ratio—in particular, for a given ERGM parameter configuration $\boldsymbol{\theta}'$—it makes sense to use a proposal distribution from the same family, with a similar $\boldsymbol{\theta}$, which, for an exponential family in particular, simplifies as follows:

$$\frac{\kappa_g(\boldsymbol{\theta}')}{\kappa_g(\boldsymbol{\theta})} = \frac{\sum\limits_{y\in\mathcal{Y}} \exp\{\boldsymbol{\theta}'\cdot g(y)\}}{\kappa_g(\boldsymbol{\theta})} = \frac{\sum\limits_{y\in\mathcal{Y}} \exp\{(\boldsymbol{\theta}'-\boldsymbol{\theta})\cdot g(y)\} \exp\{\boldsymbol{\theta}\cdot g(y)\}}{\kappa_g(\boldsymbol{\theta})}$$

$$= \sum_{y\in\mathcal{Y}} \exp\{(\boldsymbol{\theta}'-\boldsymbol{\theta})\cdot g(y)\} \frac{\exp\{\boldsymbol{\theta}\cdot g(y)\}}{\kappa_g(\boldsymbol{\theta})}. \tag{A2}$$

Now, $\exp\{\boldsymbol{\theta}\cdot g(y)\}/\kappa_g(\boldsymbol{\theta})$ is just $\Pr_{\boldsymbol{\theta};g}(Y=y)$, making equation (A2) an expectation of $\exp\{(\boldsymbol{\theta}'-\boldsymbol{\theta})\cdot g(Y)\}$ under $Y\sim\Pr_{\boldsymbol{\theta};g}(\cdot)$, so the ratio can be approximated simply with

$$\frac{\kappa_g(\boldsymbol{\theta}')}{\kappa_g(\boldsymbol{\theta})} = \mathbf{E}_{\boldsymbol{\theta};g}[\exp\{(\boldsymbol{\theta}'-\boldsymbol{\theta})\cdot g(Y)\}] \approx \frac{1}{S}\sum_{s=1}^{S} \exp\{(\boldsymbol{\theta}'-\boldsymbol{\theta})\cdot g(Y^{(s)})\}, \tag{A3}$$

given a sample $Y^{(1)}, \ldots, Y^{(S)}$ drawn from $\Pr_{\boldsymbol{\theta};g}(\cdot)$ for a given guess $\boldsymbol{\theta}$. The accuracy of approximation (A3) decreases as $\boldsymbol{\theta}'$ draws farther from $\boldsymbol{\theta}$, because the greater the difference between them, the greater the variance of the quantity in the exponent of equation (A3); so, in practice, a starting guess $\boldsymbol{\theta}^{(0)}$ is selected, sampled from, used to maximize equation (A1) to produce a new

guess $\boldsymbol{\theta}^{(1)}$, which is in turn used to draw a new sample, and so on until convergence.

Notice that equation (A3) depends on $\boldsymbol{Y}^{(s)}$ only through its vector of sufficient statistics $\boldsymbol{g}(\boldsymbol{Y}^{(s)})$. This means that once the sample is drawn, the maximization is agnostic to the nature of the data and to the structure of the sample space $\mathcal{Y}$, depending only on the sufficient statistics. In practice, this reduces its storage requirements and greatly simplifies convergence and other diagnostics, and it means that developments such as those of Hummel et al. (2012) to improve stability and accuracy of the estimation can be applied directly. We use them here as well.

To obtain such a sample, we use a Metropolis sampling algorithm (Algorithm A1). At every iteration, an ego is chosen at random and a symmetric proposal to swap the rankings of two of its alters (also chosen at random) is made. Any permutation of $n - 1$ alters for each ego can be reached from any other permutation in at most $n - 2$ swaps, and the sample space is finite, guaranteeing ergodicity.

Algorithm A1 Sampling from a Complete Rank ERGM

Let:

    RandomChoose(A) return a random element of a set A

    Uniform(a, b) return a random draw from the Uniform(a, b) distribution

Require: $\boldsymbol{y}^{(0)} \in \mathcal{Y}$, S sufficiently large, N, $\boldsymbol{g}(\,\cdot\,)$, $\boldsymbol{\theta}$
Ensure: a draw from a complete ranking ERGM $\Pr_{\boldsymbol{\theta};\boldsymbol{g}}(\,\cdot\,)$

  1: **for** $s \leftarrow \{1 .. S\}$ **do**
  2:     $i \leftarrow$ RandomChoose(N) {Select an ego at random.}
  3:     $j \leftarrow$ RandomChoose($N\backslash\{i\}$) {Propose one alter.}
  4:     $j' \leftarrow$ RandomChoose($N\backslash\{i,j\}$) {Propose another alter.}
  5:     $\boldsymbol{y}^* \leftarrow \left(\boldsymbol{y}^{(s-1)}\right)^{i:j\rightleftarrows j'}$ {Propose a swap.}
  6:     $r \leftarrow \exp\left[\boldsymbol{\theta} \cdot \{\boldsymbol{g}(\boldsymbol{y}^*) - \boldsymbol{g}(\boldsymbol{y}^{(s-1)})\}\right]$
  7:     $u \leftarrow$ Uniform($0, 1$)
  8:     **if** $u < r$ **then**
  9:         $\boldsymbol{y}^{(s)} \leftarrow \boldsymbol{y}^*$ {Accept the proposal.}
10:     **else**
11:         $\boldsymbol{y}^{(s)} \leftarrow \boldsymbol{y}^{(s-1)}$ {Reject the proposal.}
12:     **return** $\boldsymbol{y}^{(S)}$

A.3. *Degeneracy.* Degeneracy, in its many meanings, is often a concern in applying ERGMs. Despite recent progress (e.g., Butts 2011; Rinaldo, Fienberg,

and Zhou 2009; Schweinberger 2011), few general and unambiguous diagnostics for degeneracy exist beyond simulated goodness-of-fit measures (Hunter, Goodreau, and Handcock 2008) and diagnostics of symptoms such as poor convergence of the MLE-finding procedure and, in particular, poor mixing of the MCMC sampler (Goodreau et al. 2008b), often because of multimodality of the distribution of networks under the model (Krivitsky 2012; Rinaldo et al. 2009; Snijders et al. 2006).

In our case, there is some cause for concern because of the high order of some of the sufficient statistics we propose. For example, g_{GNC} is a summation of indicators over a set of cardinality $|N^{4 \neq}| = O(n^4)$. Its promotion statistic $\Delta_{i,j}^{\nearrow} g_{GNC}$ is a sum over a set of $O(n)$. Thus, there may exist configurations for which a small change in the ranking leads to a massive change in the value of this statistic and thus in the likelihood. Such statistics can induce excessively strong dependence and lead to asymptotic degeneracy (Butts 2011; Schweinberger 2011).

At the same time, a number of factors are likely to ameliorate such problems for the specific case of rank data. A network of ranks simply contains more information than a network of binary outcomes of the same size. More concretely, a uniform distribution over binary directed networks of size n with no self-loops has entropy of $n(n-1)$ bits, while a uniform distribution over rank networks has entropy of $\log_2 \{(n-1)!\}^n = n \log_2 \{(n-1)!\}$ bits, which grows much faster in n. This can reduce the risk for pathological model behavior due to poor identification in the small-n case. Also, although there exists the potential for problematic statistics and configurations, the constrained nature of the sample space makes degeneracy more difficult to achieve than is the case for typical graphs or digraphs: an ego cannot promote one alter without demoting another, and while some statistics, such as g_{GNC}, have an extreme case of all actors agreeing, others, such as g_D, cannot be reduced to 0.

In all of the examples, we found no symptoms of degeneracy: for every fit reported, the estimation algorithm had converged without issues to an estimate for which the expectation of the sufficient statistic from the penultimate MCMC MLE step was close to its observed value, MCMC diagnostics (found in the Online Supplement) showed no sign of poor mixing or multimodality, and the MCMC sample sizes used were sufficient to render the error due to the use of MCMC to approximate the likelihood (Hunter and Handcock 2006, equation 3.11) negligible compared with the standard error. That these MLEs were found and converged to despite the **0** starting point, again, suggests that models using the terms introduced here are quite stable. In practice, we

recommend simulation from fitted models (as in the case of binary ERGMs) as a useful tool for verifying nondegeneracy.

APPENDIX B

Change Statistic Examples

Section 3 introduces a variety of terms for ordinal relational data, all of which are defined in terms of the comparison relation of Section 2.1. To further fix ideas (and to provide a resource for those seeking to implement these terms), we here provide numerical examples of change statistic calculations for several of these terms. All examples are based on the illustrative ranking structure of Section 3.

B.1. *Attractiveness/Popularity Effects.* As an illustrative example of attractiveness, we consider these statistics in the cases of the sample complete-ordering networks y and $y^{A:B\rightleftharpoons C}$ from Figure 2. For purposes of illustration, we also assume that each sample network is associated with a covariate vector $x = [1, 2, 3, 4]$. Note that the attractiveness statistic g_A is equal to the sum of the signed differences in the elements of x for all those ordered pairs having nonzero entries in $y_{\cdot:\cdot\succ\cdot}$. These entries are depicted in Figure 2; summing the associated differences gives us $g_A(y; x) = 6$ and $g_A(y^{A:B\rightleftharpoons C}; x) = 4$. Intuitively, the ratings of y are more strongly aligned with the attribute values in x than those of $y^{A:B\rightleftharpoons C}$. Note that the difference between $g_A(y; x)$ and $g_A(y^{A:B\rightleftharpoons C}; x)$ is necessarily equal to the promotion statistic for a swap of B and C in A's rating structure; that is, $2(x_B - x_C) = 2(2 - 3) = -2$. In the case of attractiveness, we thus have the intuitive notion that promoting an actor with a lower attribute value over one whose attribute value is higher results in a decline in the associated statistic.

B.2. *Difference/Similarity Effects.* Returning to our two sample cases, let us consider how the distance/similarity statistic g_H varies (again taking $x = [1, 2, 3, 4]$, and for simplicity taking z equal to the absolute difference). In the case of g_A, our statistic reduced to the sum of attribute value differences for ordered pairs having a judgment in $y_{\cdot:\cdot\succ\cdot}$; the present case can be recognized as essentially similar, with the only distinction being that the relevant attribute for $y_{i:j\succ k}$ is now the proximity to the rater (i) vis-a-vis x. Summing over the ordinal judgments in each case gives us $g_H(y; x) = 6$, $g_H(y^{A:B\rightleftharpoons C}; x) = 4$, and $\Delta_{A,B}^{\nearrow} g_H(y) = 2(|x_A - x_B| - |x_A - x_C|) = 2(|1 - 2| - |1 - 3|) = -2$.

B.3. *Global Conformity.* For our sample networks, y and $y^{A:B \rightleftarrows C}$, summing over all tetrads gives us respectively $g_A(y) = 8$ and $g_A(y^{A:B \rightleftarrows C}) = 6$. As this implies, $\Delta'_{A,B} g_{GNC}(y) = 2 \sum_{l \in N \setminus \{A, B, C\}} (y_{l:C \succ B} - y_{l:B \succ C}) = 2(y_{D:C \succ B} - y_{D:B \succ C}) = 2(0 - 1) = -2$. Intuitively, this implies that the ratings of $y^{A:B \rightleftarrows C}$ are in greater consistency with one another than those of y; as the promotion statistic indicates, this is because A's switching from $C \succ_A B$ to $B \succ_A C$ brings A view of B and C into conformity with the view of D. To the extent that greater conformity pressure is present (i.e., a negative coefficient associated with g_A), such changes are (*ceteris paribus*) favored.

B.4. *Local Conformity.* For our sample networks y and $y^{A:B \rightleftarrows C}$, calculation of the local conformity statistic shows that y exhibits a slightly higher level of nonconformity than $y^{A:B \rightleftarrows C}$: $g_{LNC}(y) = 2$, versus $g_{LNC}(y^{A:B \rightleftarrows C}) = 1$. To appreciate the origin of this difference, we may examine the promotion statistic:

$$\Delta'_{A,B} g_{LNC}(y) = (y_{A:D \succ C} y_{D:C \succ B} - y_{A:D \succ C} y_{D:B \succ C}$$

$$+ y_{D:A \succ C} y_{D:C \succ B} - y_{D:A \succ B} y_{D:B \succ C}$$

$$+ y_{A:C \succ D} y_{B:D \succ C} - y_{A:B \succ D} y_{C:B \succ D})$$

$$= (1 \times 0 - 1 \times 1$$

$$+ 0 \times 0 - 0 \times 1$$

$$+ 0 \times 1 - 0 \times 1) = -1.$$

In promoting B over C, A conformed with D—the only actor A ranked over both B and C—so local nonconformity declined (per line 10a). D did not rank A highly enough to experience conformity pressure in ranking B and C, and the only actor who ranked A highly is C, but because actors are not permitted to rank themselves, C has nothing to say about C's own ranking relative to B (see line 10b). Promoting C over B made B salient in comparisons about C, but not about D, so no disconformance was created or eliminated in this respect (see line 10c). As this case illustrates, local nonconformity involves both consistency of ratings and status: to the extent that pressure for local conformity is present, actors will seek to adapt their rankings to match those of individuals they rate highly, with less regard to mismatches with those to whom they assign a low rank.

B.5. *Deference Aversion.* Although we saw that $y^{\text{A: B}\rightleftharpoons\text{C}}$ showed lower local nonconformity than y, we see the opposite for deference aversion: Applying the above definition to the two networks gives us $g_D(y) = 6$ and $g_D(y^{\text{A: B}\rightleftharpoons\text{C}}) = 8$, implying that y has fewer instances in which an individual tacitly places himself or herself below others in the network. Examining the promotion statistic, we see that

$$\Delta^{\nearrow}_{\text{A, B}} g_D(y) = 2(y_{\text{C: A}\succ\text{B}} + y_{\text{B: C}\succ\text{A}} - 1)$$

$$= 2 \times (1 + 1 - 1) = 2.$$

Intuitively, C had ranked A over B; so in ranking C over B, A was not deferring to either party. By contrast, B had ranked C over A; so in promoting B over C, A implicitly deferred to B. In settings for which deference aversion is operative, such changes are *ceteris paribus* unfavorable.

Acknowledgments

Funding

Note

1. In formal notation, this can be written as $N^{p\neq} \equiv \{\boldsymbol{v} \in N^p : \forall_{i \in \{1 \ldots p\}} \forall_{j \in \{1 \ldots p\}} i \neq j \Rightarrow v_i \neq v_j\}$.

References

Anderson, Cameron, Sanjay Srivastava, Jennifer S. Beer, Sandra E. Spataro, and Jennifer A. Chatman. 2006. "Knowing Your Place: Self-perceptions of Status in Face-to-face Groups." *Journal of Personality and Social Psychology* 91(6): 1094–1110.

Arabie, Phipps, Scott A. Boorman, and Paul R. Levitt. 1978. "Constructing Blockmodels: How and Why." *Journal of Mathematical Psychology* 17(1):21–63.

Asch, Solomon E. 1951. "Effects of Group Pressure upon the Modification and Distortion of Judgments." Pp. 177–90 in *Groups, Leadership and Men*, edited by Harold Guetzkow. Oxford, UK: Carnegie Press.

Barndorff-Nielsen, Ole E. 1978. *Information and Exponential Families in Statistical Theory*. New York: John Wiley.

Berger, Joseph, Bernard P. Cohen, and Morris Zelditch. 1972. "Status Characteristics and Social Interaction." *American Sociological Review* 37(3):241–55.

Berger, Joseph, M. Hamit Fisek, Robert Z. Norman, and Morris Zelditch. 1977. *Status Characteristics and Social Interaction: An Expectation States Approach*. New York: Elsevier.

Bernard, H. Russell, Peter Killworth, David Kronenfeld, and Lee Sailer. 1984. "The Problem of Informant Accuracy: The Validity of Retrospective Data." *Annual Review of Anthropology* 13:495–517.

Bernard, H. Russell, and Peter D. Killworth. 1977. "Informant Accuracy in Social Network Data II." *Human Communication Research* 4(1):3–18.

Bordieu, Pierre. 1968. "Outline of a Sociological Theory of Art Perception." *International Social Science Journal* 20(4):589–612.

Breiger, Ronald L., Scott A. Boorman, and Phipps Arabie. 1975. "An Algorithm for Clustering Relational Data with Applications to Social Network Analysis and Comparison with Multidimensional Scaling." *Journal of Mathematical Psychology* 12:323–83.

Brown, Lawrence D. 1986. *Fundamentals of Statistical Exponential Families with Applications in Statistical Decision Theory*. Vol. 9, Lecture Notes—Monograph Series. Hayward, CA: Institute of Mathematical Statistics.

Butts, Carter T. 2007a. "Models for Generalized Location Systems." Pp. 283–348 in *Sociological Methodology*, Vol. 37, edited by Yu Xie. Hoboken, NJ: Wiley-Blackwell.

Butts, Carter T. 2007b. "Permutation Models for Relational Data." Pp. 257–81 in *Sociological Methodology*, Vol. 37, edited by Yu Xie. Hoboken, NJ: Wiley-Blackwell.

Butts, Carter T. 2011. "Bernoulli Graph Bounds for General Random Graphs." Pp. 299–345 in *Sociological Methodology*, Vol. 41, edited by Tim Futing Liao. Hoboken, NJ: Wiley-Blackwell.

Caimo, Alberto, and Nial Friel. 2011. "Bayesian Inference for Exponential Random Graph Models." *Social Networks* 33(1):41–55.

Cliff, Andrew D., and J. Keith Ord. 1973. *Spatial Autocorrelation*. London: Pion.

Dekker, David, David Krackhardt, and Tom A. B. Snijders. 2007. "Sensitivity of MRQAP Tests to Collinearity and Autocorrelation Conditions." *Psychometrika* 72(4):563–81.

Doreian, P. 1990. "Network Autocorrelation Models: Problems and Prospects." Pp. 369–89 in *Spatial Statistics: Past, Present, and Future*, edited by I. D. A. Griffith. Ann Arbor, MI: Institute of Mathematical Geography.

Doreian, Patrick, Roman Kapuscinski, David Krackhardt, and Janusz Szczypula. 1996. "A Brief History of Balance through Time." *Journal of Mathematical Sociology* 21(1–2):113–31.

Dunbar, Robin. 1997. *Grooming, Gossip, and the Evolution of Language*. Cambridge, MA: Harvard University Press.

Enelow, James M., and Melvin J. Hinich. 1984. *The Spatial Theory of Voting: An Introduction*. Cambridge, UK: Cambridge University Press.

Geyer, Charles J., and Elizabeth A. Thompson. 1992. "Constrained Monte Carlo Maximum Likelihood for Dependent Data." *Journal of the Royal Statistical Society, Series B* 54(3):657–99.

Goodreau, Steven M., James Kitts, and Martina Morris. 2008a[b]. "Birds of a Feather, or Friend of a Friend? Using Exponential Random Graph Models to Investigate Adolescent Social Networks." *Demography* 46(1):103–25.

Goodreau, Steven M., Mark S. Handcock, David R. Hunter, Carter T. Butts, and Martina Morris. 2008b[a]. "A Statnet Tutorial." *Journal of Statistical Software* 24(1):1–26.

Gormley, Isobel Claire, and Thomas Brendan Murphy. 2008. "Exploring Voting Blocs within the Irish Electorate: A Mixture Modeling Approach." *Journal of the American Statistical Association* 103(483):1014–27.

Handcock, Mark S., and Krista J. Gile. 2010. "Modeling Social Networks from Sampled Data." *Annals of Applied Statistics* 4(1):5–25.

Handcock, Mark S., David R. Hunter, Carter T. Butts, Steven M. Goodreau, Pavel N. Krivitsky, and Martina Morris. 2014. "ergm: Fit, Simulate and Diagnose Exponential-family Models for Networks." The Statnet Project. R Package Version 3.2.4. Retrieved (http://www.statnet.org).

Hanneke, Steve, Wenjie Fu, and Eric P. Xing. 2010. "Discrete Temporal Models of Social Networks." *Electronic Journal of Statistics* 4:585–605.

Harris, Kathleen M., F. Florey, Joyce Tabor, Peter S. Bearman, J. Jones, and J. Richard Udry. 2003. "The National Longitudinal Study of Adolescent Health: Research Design." Technical report, University of North Carolina.

Holland, Paul W., and Samuel Leinhardt. 1981. "An Exponential Family of Probability Distributions for Directed Graphs." *Journal of the American Statistical Association* 76(373):33–65.

Hubert, Lawrence J. 1987. *Assignment Methods in Combinatorial Data Analysis*. New York: Marcel Dekker.

Hummel, Ruth M., David R. Hunter, and Mark S. Handcock. 2012. "Improving Simulation-based Algorithms for Fitting ERGMs." *Journal of Computational and Graphical Statistics* 21(4):920–39.

Hunter, David R., Steven M. Goodreau, and Mark S. Handcock. 2008. "Goodness of Fit for Social Network Models." *Journal of the American Statistical Association* 103: 248–58.

Hunter, David R., and Mark S. Handcock. 2006. "Inference in Curved Exponential Family Models for Networks." *Journal of Computational and Graphical Statistics* 15(3):565–83.

Hunter, David R., Mark S. Handcock, Carter T. Butts, Steven M. Goodreau, and Martina Morris. 2008. "ergm: A Package to Fit, Simulate and Diagnose Exponential-Family Models for Networks." *Journal of Statistical Software* 24(1):1–29.

Johnson, Jeffrey C., James S. Boster, and Lawrence A. Palinkas. 2003. "Social Roles and the Evolution of Networks." *Journal of Mathematical Sociology* 27(1):89–121.

Krackhardt, David. 1987. "QAP Partialling as a Test of Spuriousness." *Social Networks* 9(2):171–86.

Krackhardt, David, and Mark S. Handcock. 2007. "Heider versus Simmel: Emergent Features in Dynamic Structures." Pp. 14Ậ27 In *Statistical Network Analysis: Models, Issues, and New Directions: ICML 2006 Workshop on Statistical Network Analysis, Pittsburgh, PA, USA, June 29, 2006, Revised Selected Papers*, edited by Edoardo Airoldi, David M. Blei, Stephen E. Fienberg, Anna Goldenberg, Eric P. Xing, and Alice X. Zheng. New York: Springer.

Krivitsky, Pavel N. 2012. "Exponential-family Random Graph Models for Valued Networks." *Electronic Journal of Statistics* 6:1100–28.

Krivitsky, Pavel N., and Mark S. Handcock. 2014. "A Separable Model for Dynamic Networks." *Journal of the Royal Statistical Society, Series B* 76(1):29–46.

Levine, Joel H. 1993. *Exceptions Are the Rule: Inquiries on Method in the Social Sciences*. Boulder, CO: Westview.

Morse, Stanley J., Harry T. Reis, Joan Gruzen, and Ellen Wolff. 1974. "The Eye of the Beholder: Determinants of Physical Attractiveness Judgments in the U.S. and South Africa." *Journal of Personality* 42(4):528–42.

Nakao, Keiko, and A. Kimball Romney. 1993. "Longitudinal Approach to Subgroup Formation: Re-analysis of Newcomb's Fraternity Data." *Social Networks* 15(2): 109–31.

Newcomb, Theodore M. 1956. "The Prediction of Interpersonal Attraction." *American Psychologist* 11(11):575–86.

Newcomb, Theodore M. 1961. *The Acquaintance Process*. New York: Holt, Rinehart.

Pattison, Philippa E. 1982. "The Analysis of Semigroups of Multirelational Systems." *Journal of Mathematical Psychology* 25(1):87–118.

Plackett, Robin L. 1975. "The Analysis of Permutations." *Journal of the Royal Statistical Society, Series C (Applied Statistics)* 24(2):193–202.

R Core Team. 2014. *R: A Language and Environment for Statistical Computing*. Vienna, Austria: R Foundation for Statistical Computing.

Rinaldo, Alessandro, Stephen E. Fienberg, and Yi Zhou. 2009. "On the Geometry of Discrete Exponential Families with Application to Exponential Random Graph Models." *Electronic Journal of Statistics* 3:446–84.

Robins, Garry, Philippa Pattison, and Stanley S. Wasserman. 1999. "Logit Models and Logistic Regressions for Social Networks: III. Valued Relations." *Psychometrika* 64(3):371–94.

Sampson, Samuel F. 1968. "A Novitiate in a Period of Change: An Experimental and Case Study of Social Relationships." PhD dissertation, Department of Sociology, Cornell University.

Schweinberger, Michael. 2011. "Instability, Sensitivity, and Degeneracy of Discrete Exponential Families." *Journal of the American Statistical Association* 106(496): 1361–70.

Snijders, Tom A. B. 1996. "Stochastic Actor-oriented Models for Network Change." *Journal of Mathematical Sociology* 21(1–2):149–72.

Snijders, Tom A. B., Philippa E. Pattison, Garry L. Robins, and Mark S. Handcock. 2006. "New Specifications for Exponential Random Graph Models." Pp. 99–153 in

Sociological Methodology, Vol. 36, edited by Ross M. Stolzenberg. Boston, MA: Blackwell.

Strauss, David, and Michael Ikeda. 1990. "Pseudolikelihood Estimation for Social Networks." *Journal of the American Statistical Association* 85(409):204–12.

Thomas, Andrew C., and Joseph K. Blitzstein. 2011. "Valued Ties Tell Fewer Lies: Why Not to Dichotomize Network Edges with Thresholds." Retrieved February 3, 2017 (https://arxiv.org/pdf/1101.0788.pdf).

van Duijn, Marijtje A. J., Tom A. B. Snijders, and Bonne J. H. Zijlstra. 2004. "p2: A Random Effects Model with Covariates for Directed Graphs." *Statistica Neerlandica* 58(2):234–54.

Wasserman, Stanley. 1980. "Analyzing Social Networks as Stochastic Processes." *Journal of the American Statistical Association* 75(370):280–94.

Wasserman, Stanley S., and Philippa Pattison. 1996. "Logit Models and Logistic Regressions for Social Networks: I. An Introduction to Markov Graphs and p*." *Psychometrika* 61(3):401–25.

Webster, Murray, and James E. Driskell. 1983. "Beauty as Status." *American Journal of Sociology* 89(1):140–65.

White, Harrison C., Scott A. Boorman, and Ronald L. Breiger. 1976. "Social Structure from Multiple Networks. I. Blockmodels of Roles and Positions." *American Journal of Sociology* 81(4):730–80.

Author Biographies

Pavel N. Krivitsky is a lecturer in statistics at the School of Mathematics and Applied Statistics and the National Institute for Applied Statistics Research Australia at the University of Wollongong. His current interests focus on statistical models for social network data and processes and their formulation, implementation, and application.

Carter T. Butts is a professor in the Departments of Sociology, Statistics, and Electrical Engineering and Computer Science at the University of California, Irvine, where he directs the Center for Networks and Relational Analysis. His current research interests include statistical models for network structure and dynamics, models for interaction processes, and the response of social systems to exogenous shocks.